Results in Physics 4 (2014) 189–197

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.journals.elsevier.com/results-in-physics

Quantum state revivals in quantum walks on cycles

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ARTICLE INFO

Article history:
Received 7 June 2014
Accepted 7 October 2014
Available online 14 October 2014

Keywords:
Quantum walk
Quantum state revival
Circulant matrix
De Moivre numbers

ABSTRACT

Recurrence in the classical random walk is well known and described by the Pólya number. For quantum walks, recurrence is similarly understood in terms of the probability of a localized quantum walker to return to its origin. Under certain circumstances the quantum walker may also return to an arbitrary initial quantum state in a finite number of steps. Quantum state revivals in quantum walks on cycles using coin operators which are constant in time and uniform across the path have been described before but only incompletely. In this paper we find the general conditions for which full-quantum state revival will occur.

1. Introduction

A quantum walk is the quantum-mechanical complement to the classical random walk. In a quantum walk the "walker" evolves according to a unitary transformation between initial and final states, either in discrete steps of time or by a continuous-time evolution under a Hamiltonian operator. The discrete-time quantum walk was first described by Aharonov et al. [1] where it was noted that due to quantum interference effects the average path length of the quantum walk can be longer than the maximum allowed path length in a classical random walk. To take advantage of this phenomenon the quantum walk has since been applied to the development of quantum search algorithms [2–5] in terms of both the discrete-time [6,7] and continuous-time [8,9] quantum walks. Both the discrete and continuous quantum walks have also been shown to be universal for quantum computation [10–12].

An important problem in the study of classical random walks is determining the probability of the walker returning to its origin. This is referred to as recurrence and is determined by the random walk's Pólya number [13–15]. Recurrence in a quantum walk is similarly defined as the probability after N steps for observing the quantum walker at its point of origin [16–18]. Recurrence in continuous-time quantum walks has also been studied [19].

The criterion of the quantum walker returning to its initial quantum state or quantum state revival is a more stringent requirement. Previous work has looked at full revivals in quantum walks in a 2 dimensional graph [20]. A similar problem looking at quantum diffusion on a cyclic lattice is also treated [21]. This paper is concerned with the conditions under which a quantum walker in an arbitrary quantum state on a k-cycle with k unitary transformation sites, will return to its initial quantum state in N steps. Our assumptions are that each of the unitary transformations be time independent and equal. Quantum state revivals occur when the k-cycle operator, $U_k(\rho,\alpha,\beta)$ satisfies $U_k^N = I_{2k}$ where I_{2k} is the $2k \times 2k$ identity matrix.

2. Discrete quantum walks in one dimension

The necessary elements of the classical random walk are a walker and a random coin toss mechanism. For each toss of the coin the walker takes a step to the right if "heads" or a step to the left if "tails". An important distinction of the quantum walk is the quantum property of superposition, in this case a superposition of the amplitudes corresponding to a step to the left and a step to the right. Thus the quantum counterpart to the classical random walk involves a quantum walker with a two state coin space and a unitary coin operator. The coin operator can be continuously tuned in both how much it rotates the original state and its relative phase change. The essential quantum behavior is typically modeled in terms of a quantum two state system such as a spin $\frac{1}{2}$ particle for the walker and a general 2×2 unitary transformation matrix for the coin operator. It is possible that the coin operator may change with time or have different coin operators at each discrete position of the walk [22–24]. In all that follows, however, we will only consider a coin operator which is constant in time and uniform for all positions.

2.1. Discrete quantum walks on a line

For concreteness, we consider a two state quantum walker located at the origin of a line extending in the positive and negative

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http://dx.doi.org/10.1016/j.rinp.2014.10.001
2211-3797/

directions. Each step of the walker has equal length and occurs at discrete time intervals. Let $\mathbf{H}z$ represent the Hilbert space of the locations of the walker along the infinite line. This space is spanned by the basis states $\{|i\rangle : i \in \mathbb{Z}\}$, such that $|i\rangle$ corresponds with a walker localized at position i on the line. The coin space $\mathbf{H}c$ of our quantum walker will be spanned by basis states $\{|\uparrow\rangle, |\downarrow\rangle\}$. The Hilbert space for the walker system will now be $\mathbf{H}w = \mathbf{H}z \otimes \mathbf{H}c$, the tensor product of the position space with the coin space. In our model the spin-up and spin-down amplitudes will step in opposite directions along the line such that

$$|i,\uparrow\rangle \longrightarrow |i-1,\uparrow\rangle \tag{1a}$$
$$|i,\downarrow\rangle \longrightarrow |i+1,\downarrow\rangle \tag{1b}$$

a spin-up amplitude steps in the negative direction (to the left) and a spin-down amplitude steps in the positive direction (to the right). The conditional shift operator in $\mathbf{H}w$ which does this is

$$S_Z = \sum_{i=-\infty}^{\infty} |i-1\rangle\langle i| \otimes |\uparrow\rangle\langle\uparrow| + \sum_{i=-\infty}^{\infty} |i+1\rangle\langle i| \otimes |\downarrow\rangle\langle\downarrow| \tag{2a}$$
$$= \sum_{s=0}^{1}\sum_{i=-\infty}^{\infty} |i+2s-1\rangle\langle i| \otimes |s\rangle\langle s|, \tag{2b}$$

expression (2b) is expressed in the quantum computational basis for which $|0\rangle = |\uparrow\rangle$ and $|1\rangle = |\downarrow\rangle$.

Prior to taking each step, the coin operator would be applied to the walker's amplitude at each position $|i\rangle$ effectively rotating the spin-state into a coherent superposition of spin-up and spin-down amplitudes and thus control the portion of amplitude which is shifted to the left and to the right. A parameterization of the most general 2×2 unitary coin operator, to within a global phase change, is [24]

$$C_2 = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho}\,e^{i\alpha} \\ \sqrt{1-\rho}\,e^{i\beta} & -\sqrt{\rho}\,e^{i(\alpha+\beta)} \end{pmatrix}, \quad 0 \leqslant \rho \leq 1, \; 0 \leqslant \alpha,\beta \leqslant \pi. \tag{3}$$

The complete operator for each step of the discrete quantum walk on the infinite line is then

$$U_Z = S_Z \cdot (I_Z \otimes C_2). \tag{4}$$

This provides a uniform application of the coin operator C_2 across all the possible positions of the walker.

An often cited coin operator is the 2×2 Hadamard operator [5,11,25]

$$C_2\left(\rho = \frac{1}{2}, \alpha = 0, \beta = 0\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5}$$

Fig. 1 illustrates the four step evolution of a quantum walk on a line with the initial state $|\psi_i\rangle = |0,\uparrow\rangle$. Each step of the quantum walk is divided into a unitary transformation using the Hadamard coin operator of Eq. (5) immediately followed by the conditional shift operator in Eq. (2b).

As the walk progresses, an asymmetry in the amplitudes skewing the probabilities for finding the walker at locations on the left side of the initial position becomes evident on completion of the third step. The probabilities will be skewed to the right with an initial state of $|0,\downarrow\rangle$. The asymmetry arises from the fact that the Hadamard operator treats the two states $|\uparrow\rangle$ and $|\downarrow\rangle$ differently by inducing a phase inversion in the $|\downarrow\rangle$ amplitude. The Hadamard operator will develop a symmetric walk in the probabilities with $|\psi_i\rangle = \left(\frac{1}{\sqrt{2}}|0,\uparrow\rangle + \frac{i}{\sqrt{2}}|0,\downarrow\rangle\right)$.

2.2. Discrete quantum walks on a cycle

The conditional shift operator in Eq. (2b) can be readily modified to operate on a cycle or closed loop of k steps as

$$S_k = \sum_{s=0}^{1}\sum_{i=0}^{k-1} |i+2s-1 (\mathrm{mod}\ )k\rangle\langle i| \otimes |s\rangle\langle s| \tag{6}$$
$$U_k = S_k \cdot (I_k \otimes C_2), \tag{7}$$

where I_k is the $k\times k$ identity matrix. U_k can be expressed as a $(2k)\times(2k)$ matrix. Consider the $k=3$ loop, the operator in Eq. (7) becomes,

$$U_3 = \begin{pmatrix} 0 & 0 & \sqrt{\rho} & \sqrt{1-\rho}e^{i\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1-\rho}e^{i\beta} & -\sqrt{\rho}e^{i(\alpha+\beta)} \\ 0 & 0 & 0 & 0 & \sqrt{\rho} & \sqrt{1-\rho}e^{i\alpha} \\ \sqrt{1-\rho}e^{i\beta} & -\sqrt{\rho}e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 \\ \sqrt{\rho} & \sqrt{1-\rho}e^{i\alpha} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1-\rho}e^{i\beta} & -\sqrt{\rho}e^{i(\alpha+\beta)} & 0 & 0 \end{pmatrix}. \tag{8}$$

Fig. 2 shows a quantum walk on a $k=3$ cycle using the coin operator $C_2(\rho = \frac{2}{3},\ \alpha = 0,\ \beta = 0)$.

3. Conditions for quantum state revivals

In Fig. 2 we see it is possible, with certain choices of a constant and uniform coin operator, for the quantum walk on a cycle to return to its initial quantum state within a finite number of steps. The occurrence of quantum state revivals in quantum walks on cycles was probably first mentioned in the literature by Travaglione and Milburn [26] where they noted a revival in eight steps on a cycle with $k=4$. Later, Tregenna et.al. [24] found a handful of other instances. In this paper we wish to establish the general conditions for quantum state revivals in quantum walks on cycles. We observe that the operator U_k is a 2×2 block-circulant matrix [27,28] and is the generator of a unitary cyclic group. When the unitary operator $U_k(\rho,\alpha,\beta)$ generates a finite cyclic group, quantum state revival will occur.

3.1. Circulant matrices

If you run into a circulant in the course of a problem you are happy to make its acquaintance. – Persi Diaconis

Of the many interesting properties circulant matrices have the following are the most important for us:[29]

- The class circulant is closed under product, transpose, and inverse operations.
- All circulants are simultaneously diagonalized by the Fourier matrix.

Due to its circulant symmetries only a single row or column of a circulant matrix is required to specify it. The first row or column of a circulant matrix is referred to as its circulant vector v. For an $M\times M$ circulant matrix $A = (a_{m,n})$ the first row circulant vector is $v = (a_0, a_1, \ldots, a_{M-1})$ where we denote the first position with 0 as a matter of convenience. The circulant matrix A can then be represented as

$$A = (a_{(n-m)(\mathrm{mod}\ M)})_{m,n} = CIRC_M(a_0, a_1, \ldots, a_{M-1}). \tag{9}$$

The quantum walk operator U_k of Eq. (7) is 2×2 block circulant and is represented in terms of a circulant vector of 2×2 matrices,

$$U_k = \mathrm{CIRC}_k\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_0, \begin{bmatrix} \sqrt{\rho} & \sqrt{1-\rho}e^{i\alpha} \\ 0 & 0 \end{bmatrix}_1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_2, \ldots, \begin{bmatrix} 0 & 0 \\ \sqrt{1-\rho}e^{i\beta} & -\sqrt{\rho}e^{i(\alpha+\beta)} \end{bmatrix}_{k-1}\right). \tag{10}$$