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Invited paper

Data distributions in magnetic resonance images: A review

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A B S T R A C T

Many image processing methods applied to magnetic resonance (MR) images directly or indirectly rely on prior knowledge of the statistical data distribution that characterizes the MR data. Also, data distributions are key in many parameter estimation problems and strongly relate to the accuracy and precision with which parameters can be estimated. This review paper provides an overview of the various distributions that occur when dealing with MR data, considering both single-coil and multiple-coil acquisition systems. The paper also summarizes how knowledge of the MR data distributions can be used to construct optimal parameter estimators and answers the question as to what precision may be achieved ultimately from a particular MR image.

Introduction

Magnetic resonance imaging (MRI) is the diagnostic tool of choice in biomedicine. It is able to produce high-quality three-dimensional images containing an abundance of physiological, anatomical and functional information. A voxel's grey level within an MR image represents the amplitude of the radio frequency signal coming from the hydrogen nuclei (protons) within that voxel. To draw reliable diagnostic conclusions from MR images, visual inspection alone is often insufficient. Quantitative data analysis is required to extract the information needed. Such an analysis can almost without exception be formulated as a parameter estimation problem. The parameters of interest can simply be the values of the true MR signal underlying the noise corrupted data points [1–3], but also proton densities (in the construction of proton density maps [4,5]), relaxation time constants (in the construction of T_1, T_2 and T_2^* maps [4–11]) or diffusion parameters (in diffusion MRI) [12–14]. Different estimators can be constructed to estimate one and the same parameter, but it is well known that the best estimators (in terms of accuracy and precision) are constructed by properly taking the statistical distribution of the data into account. Hence, knowledge of the MRI data distribution is of vital importance.

This review paper gives an overview of the various distributions that occur when dealing with MR data, considering both single-coil and multiple-coil systems. The paper also summarizes how knowledge of these distributions can be used to construct optimal estimators and to answer the question as to what precision may be achieved ultimately from a particular MR image.

The organization of the paper is as follows. Section 2 briefly reviews MR signal detection and introduces a statistical model of the complex valued raw MR data acquired in the so-called *k*-space (i.e., the spatial frequency domain). Section 3 then describes the statistical distribution of the reconstructed images in the spatial domain, assuming the data have been acquired using a single-coil system. Complex images as well as magnitude and phase images, which can be constructed from the complex images straightforwardly, are considered. Since image acquisition with multiple coils is becoming more and more common nowadays, Section 4 describes the distribution of complex and magnitude images acquired with multiple-coil systems. Section 5 reviews the theory that explains how knowledge of the distribution of the MR images can be used to (i) derive a lower bound on the variance of any unbiased estimator of parameters from these images (the so-called Cramér-Rao Lower Bound), and (ii) to construct the maximum likelihood (ML) estimator, which attains this lower bound at least asymptotically. In Section 6, this theory is applied to (i) derive the CRLB for unbiased estimation of the underlying true signal amplitude from (single-coil) magnitude images and, (ii) derive the ML estimator for this estimation problem. In Section 7, conclusions are drawn.

Notation: throughout this paper, vectors will be underlined and matrices will be expressed in capital letters. Furthermore, random

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variables (RVs) will be expressed in bold face. The operators $\mathbb{E}[\cdot]$ and $Var(\cdot)$ denote the expectation and variance of a random variable, respectively. The real part of a complex valued variable z is denoted as z_R and the imaginary part as z_I. The complex conjugate of X is denoted as X^* and the transpose and complex conjugate transpose of X are denoted as X^T and X^H, respectively. Furthermore, we use $f_{\mathbf{x}}(x)$ to denote the probability density function (PDF) of the random variable $\mathbf{x}$. The conditional PDF of the RV $\mathbf{x}$ conditioned on the RV $\mathbf{y}$ is denoted as $f_{\mathbf{x}|\mathbf{y}}(x|y)$. The modified Bessel function of the first kind of order ν is denoted as $I_\nu(\cdot)$. The symbol $\imath$ denotes $\sqrt{-1}$.

Signal detection and modeling

This section briefly reviews the mathematics behind signal detection in MRI and describes the concepts of signal demodulation and quadrature detection. The section is to a large extent based on Refs. [15–19]. For a more comprehensive description, the reader is referred to those references. The final purpose of the section is to introduce a statistical model of the detected MR signal.

Modeling the noise free signal

In MRI, an object is placed in a strong static, external, homogenous magnetic field $\underline{B}_0$ that polarizes the protons in the object, yielding a net magnetic moment oriented parallel to $\underline{B}_0$. Let's assume that $\underline{B}_0$ points in the z-direction. Next, a radio frequency pulse is applied that generates another, oscillating magnetic field $\underline{B}_1$ perpendicular to $\underline{B}_0$. This so-called excitation field tips away the net magnetic moment from the z-axis, producing a magnetization component transverse to the static field. This transverse magnetization component precesses at the so-called Larmor frequency

$$\omega_0 = \gamma\left|\underline{B}_0\right|,$$

with γ the gyromagnetic ratio. This precessing magnetization vector induces a voltage in the receiver/detector coil (a conducting loop). Spatial information can be encoded in the received signal by augmenting $\underline{B}_0$ with additional, spatially varying magnetic fields. These so-called gradient fields vary linearly in space and are denoted as G_x, G_y and G_z. For example, when G_x is applied, the strength of the static magnetic field will vary with position in the x-direction as $\left|\underline{B}_z(x)\right| = \left|\underline{B}_0\right| + G_x x$, where the subscript z is used to denote that the magnetic field points in the z-direction. In this way, gradient fields can be used to make the precession frequency vary linearly in space. MRI signal detection is based on Faraday's law of electromagnetic induction and the principle of reciprocity [15]. Assuming a static inhomogeneous magnetic field pointing in the z-direction, the (noise free) voltage signal $v(t)$ in the receiver coil is related to the transverse magnetization distribution $\underline{M}_{xy}(\underline{r},t)$ of the object by the expression [15]

$$v(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},0_+\right)\right| e^{-t/T_2\left(\underline{r}\right)} \cos\left[-\omega\left(\underline{r}\right)t + \phi_e\left(\underline{r}\right) - \phi_r\left(\underline{r}\right) + \frac{\pi}{2}\right] d\underline{r} \tag{1}$$

with $\underline{r} = (x,y,z)^T$ the position in the laboratory frame, $t = 0_+$ the time instant immediately after the excitation pulse, $\omega(\underline{r})$ the free precession frequency, T_2 a relaxation time constant, $\underline{B}_{r,xy}(\underline{r})$ the detection sensitivity of the coil, $\phi_r(\underline{r})$ the reception phase angle, and $\phi_e(\underline{r})$ the initial phase shift introduced by RF excitation. The detection sensitivity $\underline{B}_{r,xy}(\underline{r})$ is defined as the xy vector component of the field generated at $\underline{r}$ by a unit current in the coil. The phase contributions $\phi_r(\underline{r})$ and $\phi_e(\underline{r})$ take a value between 0 and 2π depending on the direction of, respectively, $\underline{B}_{r,xy}(\underline{r})$ and $\underline{M}_{xy}(\underline{r},0_+)$ in the transverse plane [15]. Assuming that a frequency encoding gradient G_x was turned on during the signal read out (i.e., during data acquisition), we have

$$\omega\left(\underline{r}\right) = \omega_0 + \Delta\omega\left(\underline{r}\right), \tag{2}$$

with

$$\Delta\omega\left(\underline{r}\right) = \gamma G_x x, \tag{3}$$

where $\Delta\omega(\underline{r})$ is the spatially varying resonance frequency in the Larmor-rotating frame, i.e., the coordinate system whose transverse plane is rotating clockwise at an angular frequency ω_0 [15]. Furthermore, if we assume that a so-called phase encoding gradient G_y was turned on for a time interval T_{pe} before the signal read out, we have to add a position dependent initial phase contribution $\phi_{pe}(\underline{r})$ to $v(t)$:

$$v(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},0_+\right)\right| e^{-t/T_2\left(\underline{r}\right)} \cos\left[-\omega\left(\underline{r}\right)t - \phi_{pe}\left(\underline{r}\right) + \phi_e\left(\underline{r}\right) - \phi_r\left(\underline{r}\right) + \frac{\pi}{2}\right] d\underline{r}, \tag{4}$$

with

$$\phi_{pe}\left(\underline{r}\right) = \gamma G_y y T_{pe}. \tag{5}$$

MR image reconstruction concerns the inverse problem of reconstructing the transverse magnetization distribution $\underline{M}_{xy}(\underline{r},t)$ from the voltage signal $v(t)$. If we assume that a slice selective gradient G_z has been applied in the z-direction during the excitation period, only protons in the selected slice (at, say, $z = z_0$) are excited, so that $\underline{M}_{xy}(x,y,z_0,t) = \underline{M}_{xy}(x,y,t)$ [18]. The MRI reconstruction problem then reduces to producing a spatial map in two dimensions. Assuming that $\left|\underline{M}_{xy}(\underline{r},0_+)\right| e^{-t/T_2(\underline{r})}$ is relatively constant during data acquisition, Eq. (4) can be simplified to

$$v(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},t_{acq}\right)\right| \cos\left[-\omega\left(\underline{r}\right)t - \phi_{pe}\left(\underline{r}\right) + \phi_e\left(\underline{r}\right) - \phi_r\left(\underline{r}\right) + \frac{\pi}{2}\right] d\underline{r} \tag{6}$$

with t_{acq} the time at the center of the acquisition and

$$\underline{M}_{xy}\left(\underline{r},t_{acq}\right) = \left|\underline{M}_{xy}\left(\underline{r},0_+\right)\right| e^{-t_{acq}/T_2\left(\underline{r}\right)} e^{\imath\phi_e\left(\underline{r}\right)}. \tag{7}$$

In practice, $\Delta\omega(\underline{r}) \ll \omega_0$ and $v(t)$ is a *high frequency* bandpass signal centered about the frequency $\pm\omega_0$. The high-frequency nature of $v(t)$ may cause unnecessary problems for electronic circuits in later processing stages [15]. In practice, these problems are circumvented by exploiting the following property of the bandpass signal $v(t)$. It can be shown that the bandpass signal $v(t)$ can be represented as [19]