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On a new generalization of Fibonacci quaternions

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ABSTRACT

In this paper, we present a new generalization of the Fibonacci quaternions that are emerged as a generalization of the best known quaternions in the literature, such as classical Fibonacci quaternions, Pell quaternions, k -Fibonacci quaternions. We give the generating function and the Binet formula for these quaternions. By using the Binet formula, we obtain some well-known results. Also, we correct some results in [3] and [4] which have been overlooked that the quaternion multiplication is non commutative.

1. Introduction

The quaternions are a number system which extends to the complex numbers. They are members of non commutative algebra, first invented by William Rowan Hamilton in 1843. A quaternion q is defined in the form

$$q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3,$$

where q_0, q_1, q_2, q_3 are real numbers and i_1, i_2, i_3 are standard orthonormal basis in $\mathbb{R}^3$ which satisfy the quaternion multiplication rules as:

$$i_1^2 = i_2^2 = i_3^2 = -1,$$
$$i_1 i_2 = i_3 = -i_2 i_1, \quad i_2 i_3 = i_1 = -i_3 i_2, \quad i_3 i_1 = i_2 = -i_1 i_3.$$

There has been an increasing interest on quaternions were encountered in various areas such as computer sciences, physics, differential geometry, quantum physics, signal, color image processing, geostatics and analysis. For a survey on quaternions we refer to [1,6,10]. In addition to this, the Fibonacci sequence is perhaps one of the most well-known sequence and it has many interesting properties and important applications to diverse disciplines such as mathematics and computer science. Several researchers that work on these two topics are interested in obtaining generating functions, Binet's formulas, several identities for defined different types of sequences of quaternions.

Horadam [11] defined the Fibonacci quaternions as

$$Q_n = F_n + F_{n+1} i_1 + F_{n+2} i_2 + F_{n+3} i_3,$$

where F_n is the nth Fibonacci number defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. Some results on Fibonacci quaternions can be found in [8,9,12,14–18,22].

There are a lot of generalizations of the Fibonacci numbers. In particular, there exist a generalization called the k-Fibonacci numbers. For any positive real number k, the k-Fibonacci sequence $\{F_{k,n}\}$, is defined by

$$F_{k,n} = kF_{k,n-1} + F_{k;n-2} \quad ; \quad n \geq 2$$

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with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. For further information about this sequence we refer to [7,13]. By considering this sequence, Ramirez [20] defined the k-Fibonacci quaternion numbers and obtained some combinatorial properties of the k -Fibonacci quaternions. Recently, Catarino [3] introduced the $h(x)$- Fibonacci quaternion polynomials that generalize the k -Fibonacci quaternion numbers. By using the Binet formula, Catarino [3] obtained the Cassini's and the Catalan's identities. But it was overlooked that the quaternion multiplication is not commutative. Therefore the results in [3, Theorems 3.8–3.10] are not true. Due to the same reason the results in [4, Theorems 7and 8] are incorrect. This is one of the main objectives that we want to point out here.

Now, we consider a new generalization of the Fibonacci numbers, named as, bi-periodic Fibonacci numbers. They are emerged as a generalization of the best known sequences in the literature, such as Fibonacci sequence, Pell sequence, k-Fibonacci sequence, etc. Edson and Yayenie [5] introduced the bi-periodic Fibonacci numbers as:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2 \tag{1.1}$$

with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero numbers. If we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence, if we take $a = b = 2$ in $\{q_n\}$, we get the Pell sequence and if we take $a = b = k$ in $\{q_n\}$, we get the k-Fibonacci sequence.

In this paper, we define the bi-periodic Fibonacci quaternions that generalize the k- Fibonacci quaternion numbers given in [3] and [20]. The outline of this paper is as follows: In the rest of this section, we introduce some necessary definitions and mathematical preliminaries, which is required; in Section 2, we introduce the bi-periodic Fibonacci quaternions and give the generating function and the Binet formula for these quaternions. By using Binet's formula we give the Cassini's identity and the Catalan's identity for even indices of these quaternion sequences. Moreover, we correct the results in [3, Theorem 3.9] and [4, Theorem 7].

We start by recalling some basic results concerning quaternion algebra $\mathbf{H}$ and the bi-periodic Fibonacci numbers.

It is well known that the algebra $\mathbf{H} = \{a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 : a_i \in \mathbb{R}, i = 0, 1, 2, 3\} \cong \mathbb{C}^2$ of real quaternions is defined as the four-dimensional vector space over $\mathbb{R}$ having a basis $e_0 \cong 1$, $e_1 \cong i_1$, $e_2 \cong i_2$ and $e_3 \cong i_3$, which satisfies the following multiplication rules:

$$e_l^2 = -1, \quad l \in \{1, 2, 3\}; \quad e_1e_2 = -e_2e_1 = e_3,$$
$$e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2. \tag{1.2}$$

The bi-periodic Fibonacci sequence $\{q_n\}$ satisfy the following recurrence:

$$q_n = (ab + 2)q_{n-2} - q_{n-4}, \quad n \geq 4. \tag{1.3}$$

The generating function of the sequence $\{q_n\}$ is

$$F(x) = \frac{x + ax^2 - x^3}{1 - (ab + 2)x^2 + x^4} \tag{1.4}$$

and the Binet formula of the sequence $\{q_n\}$ is given by

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{1.5}$$

where $\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}$ that is, α and β are the roots of the polynomial $x^2 - abx - ab$ and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. It can be shown that

$$q_{-n} = (-1)^{n-1} q_n. \tag{1.6}$$

For further information about these sequences see [5,19,21,23].

2. The bi-periodic Fibonacci quaternions

Definition 1. The bi-periodic Fibonacci quaternions $\{Q_n\}$ are defined by

$$Q_n = \sum_{l=0}^{3} q_{n+l} e_l, \tag{2.1}$$

where q_n is the nth bi-periodic Fibonacci number.

Note that, if we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci quaternion numbers (see [9]), if we take $a = b = 2$, we get the Pell quaternion numbers (see [4]) and if we take $a = b = k$, we get the k-Fibonacci quaternion numbers (see [3,20]).

Theorem 1. *The generating function for the bi-periodic Fibonacci quaternion Q_n is*

$$G(t) = \frac{Q_0 + (Q_1 - bQ_0)t + (a - b)R(t)}{1 - bt - t^2} \tag{2.2}$$

where

$$R(t) := tf(t)e_0 + (f(t) - t)e_1 + \left(\frac{f(t)}{t} - 1 \right) e_2 + \left(\frac{f(t) - \left(t + (ab+1)t^3\right)}{t^2} \right) e_3, \tag{2.3}$$

$$f(t) := \sum_{n=1}^{\infty} q_{2n-1} t^{2n-1} = \frac{t - t^3}{1 - (ab + 2)t^2 + t^4}. \tag{2.4}$$

Proof. We begin with the formal power series representation of the generating function for $\{Q_n\}$,

$$G(t) = \sum_{n=0}^{\infty} Q_n t^n = Q_0 + Q_1 t + Q_2 t^2 + \cdots + Q_n t^n + \cdots.$$

Note that,

$$btG(t) = bQ_0 t + bQ_1 t^2 + \cdots + bQ_n t^{n+1} + \cdots$$

and

$$t^2 G(t) = Q_0 t^2 + Q_1 t^3 + \cdots + Q_n t^{n+2} + \cdots.$$

Since $q_{2k+1} = bq_{2k} + q_{2k-1}$ and $q_{2k} = aq_{2k-1} + q_{2k-2}$, we get

$$\begin{aligned} \left(1 - bt - t^2\right)G(t) &= Q_0 + (Q_1 - bQ_0)t \\ &\quad + \sum_{n=2}^{\infty} (Q_n - bQ_{n-1} - Q_{n-2})t^n \\ &= Q_0 + (Q_1 - bQ_0)t \\ &\quad + (a - b)t \left(\sum_{n=1}^{\infty} q_{2n-1} t^{2n} \right) e_0 \end{aligned}$$