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Portfolio selection problem with liquidity constraints under non-extensive statistical mechanics

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ABSTRACT

In this study, we consider the optimal portfolio selection problem with liquidity limits. A portfolio selection model is proposed in which the risky asset price is driven by the process based on non-extensive statistical mechanics instead of the classic Wiener process. Using dynamic programming and Lagrange multiplier methods, we obtain the optimal policy and value function. Moreover, the numerical results indicate that this model is considerably different from the model based on the classic Wiener process, the optimal strategy is affected by the non-extensive parameter q, the increase in the investment in the risky asset is faster at a larger parameter q and the increase in wealth is similar.

1. Introduction

The portfolio selection problem is an important and attractive issue in finance. Markowitz [1] was the first to consider the optimal portfolio selection problem and presented the mean-variance approach. This method is a single-period model that makes a one-off decision at the beginning of the period and holds on until the end of the period. Afterwards, Merton [2] extended this single-period model to a continuous-time model by using utility functions and the stochastic control theory.

In real financial markets, to improve risk management, agents often impose some restrictions on their trading, of which the liquidity limit has received much attention from researchers. For example, Xu and Shreve [3] studied a continuous-time portfolio selection problem with a short-selling constraint on a finite horizon and obtained the solution by solving its dual problem. Fu and Lavassani [4] obtained the explicit solutions of the dynamic mean-variance optimal portfolio selection problem with borrowing limits using the stochastic piecewise linear-quadratic control theory. Vila and Zariphopouiou [5] studied an optimal consumption and portfolio selection problem with the borrowing restriction using the stochastic dynamic programming. Luo and Wang [6] studied the portfolio selection problem that occurs when tracking the expected wealth process with liquidity limits and obtained the corresponding Hamilton-Jacobi-Bellman equation with liquidity constraints. Tepla [7] considered an optimal intertemporal portfolio problem with a borrowing limit and short-sale restrictions and introduced an algorithm for its calculation.

However, the above portfolio selection problems were modeled in the framework in which the prices of risky assets were driven by the classical Brownian motions. We know

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that this hypothesis implies that the return distributions of risky assets are normal. However, several empirical results have shown that the returns of financial quantities have the characteristics of fat-tails and aiguilles and are not normal [8–12].

In 1988, Tsallis [13] proposed the non-extensive thermostatistics, which is a generalization of the classical Boltzmann-Gibbs statistics. This non-extensive statistical mechanics was rapidly applied to various fields [14–17]. It has also increasingly drawn attention from finance researchers. For example, Queirós [18] studied characteristics of the return distributions for the Dow Jones and NYSE and found that the q-Gaussian distributions derived by the non-extensive statistics mechanics can fit the return distributions in different time scales. Borland [19,20] proposed several option pricing models under the framework of non-extensive statistics mechanics, obtaining some closed-form solutions; the models can predict option prices better than the classic Black-Scholes pricing equations. Namaki and Lai et al. [21] applied the Tsallis non-extensive statistics mechanics to detect crises of the financial markets. Stavroyiannis and Makris [22] applied the non-extensive statistics methodology to calculate the Value-at-Risk of financial time series and obtained closed-form solutions. A number of studies have shown that the non-extensive statistics methodology has been applied well to the financial field.

In this paper, we establish an optimal portfolio selection model under the framework of the non-extensive statistical mechanics and impose liquidity constraints on it.

This article is organized as follows. In Section 2, we model the price process of the risky assets by using Tsallis non-extensive statistical mechanics. The model implicates that the return distributions are q-Gaussian distributions with fat-tail characteristics rather than Gaussian ones. In Section 3, we propose an optimal portfolio selection model, which minimizes the cumulate variance between the wealth process and the expected wealth process and is subject to a liquidity constraint. In Section 4, we apply the dynamic programming methodology and Lagrange multiplier to solve our optimal portfolio selection problem. In Section 5, we present and discuss the numerical results. In the final section, we summarize the paper.

2. The risky asset price process

Recently, the empirical results have shown that the distributions of stock returns have significant fat-tail characteristics and are not normal distributions. To better fit the fat-tail characteristics of the stock return distribution, we use a stock return fluctuations model, which can be derived from the stochastic processes under the non-extensive statistical framework, to replace the standard Black-Scholes model (see [19,20]). The model is given by

$$dS(t) = S(t)(\mu dt + \sigma d\Omega) \tag{1}$$

where

$$d\Omega(t) = P(\Omega, t)^{\frac{1-q}{2}} dW(t) \tag{2}$$

$W(t)$ is a Wiener process. $P(\Omega)$ is the Tsallis distribution

$$P(\Omega, t) = \frac{1}{z(t)}(1 - \beta(t)(1-q)\Omega^2)\frac{1}{1-q} \tag{3}$$

with

$$z(t) = ((2-q)(3-q)ct)^{\frac{1}{3-q}} \tag{4}$$

$$\beta(t) = c^{\frac{1-q}{3-q}}((2-q)(3-q)t)^{\frac{-2}{3-q}} \tag{5}$$

and

$$c = \frac{\pi}{q-1}\frac{\Gamma^2(\frac{1}{q-1}-\frac{1}{2})}{\Gamma^2(\frac{1}{q-1})} \tag{6}$$

In the limit $q \to 1$, the Black-Scholes model is recovered. When $1 < q < 5/3$, the distribution exhibits power law tails and has finite variance, which covers the values of empirical returns (see [19]). Hence, this model generalizes the standard Black-Scholes model and can more accurately fit the movements of asset price.

3. Market model and liquidity constraint

Suppose there is a financial market that consists of $n+1$ assets. One is a risk-free bond whose price process $S_0(t)$ satisfies the following ordinary differential equation

$$\begin{cases} dS_0(t) = rS_0(t)dt, & t \in [0, T] \\ S_0(0) = s_0 > 0 \end{cases} \tag{7}$$

In this equation, the constant r is a positive risk-free rate. The other n assets are stocks whose price processes satisfy the model described in the second section, which is written as the following stochastic differential equation:

$$\begin{cases} dS_i(t) = S_i(t)(\mu_i dt + \sum_{j=1}^{n} \sigma_{i,j} d\Omega_j(t)), \\ \qquad i = 1, 2, \dots, n;\ t \in [0, T] \\ S_i(0) = s_i > 0 \end{cases} \tag{8}$$

where

$$d\Omega_j(t) = P(\Omega_j, t)^{\frac{1-q_j}{2}} dW_j(t) \tag{9}$$

$W_j(t), j = 1, 2, \dots, n$ is a Wiener process and $P(\Omega_j, t)$ is the Tsallis distribution of index q_j described in the second section.

Let $\mu = (\mu_1, \mu_2, \dots, \mu_n)^T$ be an R^n valued appreciation rate of returns, $\sigma = \{\sigma_{i,j}, i, j = 1, 2, \dots, n\}$ be a $n \times n$-matrix valued volatility rate of returns, and $\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_n(t))^T \in L^2_{\mathcal{F}}([0, T]; R^n)$ be a control process. The component $\pi_i(t)$ is the proportion of the investor's wealth invested in the ith risky asset $(i = 1, 2, \dots, n)$ at time t. That is, at time t, the agent's wealth can be given by

$$X^\pi(t) = \sum_{i=0}^{n} \pi_i(t) S_i(t), t \geq 0 \tag{10}$$

where

$$\pi_0(t) = 1 - \sum_{i=1}^{n} \pi_i(t)$$

Then, the wealth process $\{X^\pi(t)\}$ satisfies the following stochastic differential equation:

$$\begin{cases} dX^\pi(t) = [rX(t) + (\mu - r)^T \pi(t)]dt + \pi(t)^T \sigma d\Omega(t) \\ X(0) = x_0 > 0 \end{cases} \tag{11}$$