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Lattice fractional Laplacian and its continuum limit kernel on the finite cyclic chain

T.M. Michelitsch[a,*], B. Collet[a], A.F. Nowakowski[b,c], F.C.G.A. Nicolleau[a,b,c]

[a] Sorbonne Universités, Université Pierre et Marie Curie, Paris 6 and Institut Jean le Rond d'Alembert, CNRS UMR 7190, 4 Place Jussieu, Paris cedex 05 75252, France

[b] Department of Mechanical Engineering, Sir Frederick Mappin Building, Mappin Street, Sheffield, S1 3JD, United Kingdom

[c] Sheffield Fluid Mechanics Group, University of Sheffield, United Kingdom

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ABSTRACT

The aim of this paper is to deduce a discrete version of the fractional Laplacian in matrix form defined on the 1D periodic (cyclically closed) linear chain of finite length. We obtain explicit expressions for this fractional Laplacian matrix and deduce also its periodic continuum limit kernel. The continuum limit kernel gives an exact expression for the fractional Laplacian (Riesz fractional derivative) on the finite periodic string. In this approach we introduce two material parameters, the particle mass μ and a frequency Ω_α. The requirement of finiteness of the the total mass and total elastic energy in the continuum limit (lattice constant $h \to 0$) leads to scaling relations for the two parameters, namely $\mu \sim h$ and $\Omega_\alpha^2 \sim h^{-\alpha}$. The present approach can be generalized to define lattice fractional calculus on periodic lattices in full analogy to the usual 'continuous' fractional calculus.

1. Introduction

Many phenomena in nature are characterized by particle trajectories with irregular non-differentiable 'complex' characteristics which often appear to be 'similar' if one changes the scale. This property of self-similarity has as consequence that these phenomena cannot be described by integer order partial differential equations. However, it has turned out that application of certain nonlocal 'fractional' operators on these trajectories may be well defined and hence these phenomena can be described by fractional partial differential equations, i.e. by differential equations of non-integer orders. This inevitable change of mathematical tools to describe such 'anomalous' phenomena comes along with need of change of the geometrical description of these 'fractal' trajectories. The traditional geometrical description of integer order dimensions has to be given up and generalized to a description with non-integer order 'fractal' dimensions. The need of a 'fractionalized' description can occur with respect to space and time.

The notion of fractal geometry introduced by Mandelbrot [4,8] taught us that the classical idealizations for the trajectories of motions by smooth lines are rather rarely justified in natural structures. Instead nature chooses irregular non-differentiable self-similar curves with a fractal non-integer dimension. It is today believed that fractional calculus in his numerous variants is the appropriate mathematical tool to analyze such irregular motions.

* Corresponding author. Tel.: +33 144274928.
E-mail address: michel@lmm.jussieu.fr (T.M. Michelitsch).
URL: http://www.sheffield.ac.uk/fm (A.F. Nowakowski), http://www.sheffield.ac.uk/fm (F.C.G.A. Nicolleau)

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Once a power law occurs in Fourier space, the Fourier transformed quantity is described naturally by a fractional operator. There is recently a vast literature emerging to develop appropriate approaches to describe fractal phenomena in various physical contexts. 1D linear chains of infinite length leading to fractal dispersion relations were analyzed in [12,23]. One direction of analysis suggests to model fractal domains embedded in the physical space by vector analysis developed for spaces of non-integer dimensions [25,26], and see also the references therein. We emphasize that despite of the wide field of possible fractal applications of the approach to be developed, "fractals" are not the subject of the present paper. The present paper is devoted to develop exact representations of fractional lattice Laplacian defined on finite periodic linear chains and to the analysis of its continuum limits. An overview on the relations between fractional calculus and fractal curves such as the Weierstrass–Mandelbrot function has been presented by West [29].

Power law behavior which is naturally described by fractional calculus occurs in various completely different contexts such as anomalous and turbulent diffusion, critical phenomena such as phase transitions, biological systems, the human economy, and last but not least the present crisis of the financial system teaches us that complex systems such as the world economy does not obey Gaussian statistics where extreme events are extremely seldom, however they are governed by stable heavy tailed Lévy distributions where extreme events are in the heavy tail still rare, but they are anyway much more likely as in Gaussian cases [7].

There is a great variety of definitions for fractional integrals and derivatives depending on the function spaces in which they are defined, including Riemann, Liouville, Caputo, Grünwald–Letnikov, Marchaud, Weyl, Riesz, Feller, among others, see e.g. [2,3,6,9–11,18,20–22,24,29].

In the review article of Metzler and Klafter [9] the random walk concept is applied in order to derive fractional equations of diffusion, diffusion–advection, and Fokker–Planck type. In that article a broad overview of applications of fractional calculus on processes of anomalous diffusion is presented. In the paper of Metzler et al. [11] a fractional generalization of Fokker–Planck equation is derived by analyzing the jump rates of a generalized master equation. It can be said that the fractional approach is inevitable to describe such anomalous phenomena. In another recent review article by the same authors [10] various 'anomalous' processes are discussed which are governed by fractional evolution equations, and a broad collection of systems is presented which exhibit anomalous processes such as Lévy walks and -flights with long-range correlations. In that article both, theoretical and experimental issues of anomalous transport processes governed by fractional dynamics are analyzed.

Whereas continuum fractional calculus is well developed, the lattice fractional calculus has rarely been considered. The development of lattice fractional calculus has become an important issue due to a variety of newly emerging applications, e.g. for the description of the dynamics on networks [19]. In two recent articles by Tarasov, fractional calculus on infinite lattices has been introduced [27,28]. In [28] an analogue of vector fractional calculus is suggested. In that paper long range interparticle interaction kernels are proposed defining fractional partial derivatives on lattices with power law Fourier transforms, similar to the kernels of continuum fractional derivatives. The difference of that approach and those to be introduced in the present paper is that we deduce from 'fractional' elastic potentials the fractional Laplacian matrix on the *finite* cyclic chain. To this end we start with an elastic potential generated by a quadratic form involving a power law matrix function (below relation (8)) of the discrete Born von Karman Laplacian matrix which is defined by (7). The fractional lattice approach suggested in [27] holds on infinite bounded lattices. The approach introduced in the present paper includes both, infinite and finite cyclic chains.

The present paper aims to introduce a definition of fractional Laplacian which is analogue to the continuous version, being defined as a non-integer power of a "Laplacian", where the Laplacian is the symmetric centered second difference operator (Born-von Karman Laplacian matrix) being well defined on the periodic finite chain. A fractional generalization of the Laplacian defined on circulant networks have been analyzed by Riascos and Mateos [19] (see also the references therein). In that paper explicit results for the infinite cyclic ring have been obtained which are consistent with those deduced in the present paper. The study of Laplacian matrices generally is of importance in graph theory as it gives information on topological properties of the network [19]. In this context the fractional Laplacian matrix to be introduced is of interest for the study of anomalous 'fractional diffusion' processes taking place on the cyclic chain.

From an engineering point of view the analytical tools developed in the present paper can be of interest to improve the existing description in turbulent diffusion. Such fractional approaches could have impacts due to their capacity to describe irregular, erratic and 'turbulent' phenomena, to improve aerodynamics models and performances of aerodynamic properties in aerospace and car production engineering. Since such erratic dynamic systems cannot be described by conventional differential equations with spatial or temporal derivatives of integer orders having smooth and continuous solution characteristics. The erratic dynamical characteristics of 'turbulence trajectories' require to be described by a another language, namely the *fractional* language since fractional order derivatives of those trajectories are defined whereas their integer order derivatives do not exist [30].

In the present paper we focus on the cases where the fractional operators occur in the space domain, in the form of a fractionally generalized Laplace operator $-(-\tilde{\Delta})^{\frac{\alpha}{2}}$ with positive power law index $\alpha > 0$. In the continuum limit such systems are characterized by asymptotic $-k^{\alpha}$-power law behavior in Fourier space. An example for a discrete model which yields in the continuum limit the 1D infinite space *fractional Laplacian* (Riesz fractional derivative)[1] has been developed in recent years [12–14].

The present paper is organized as follows: first we define the discrete version of the $N \times N$-matrix of fractional Laplacian on the infinite linear chain as a limiting case of a cyclically closed (periodic) linear chain of identical $N \to \infty$ particles. N indicates the number of identical particles of the periodic chain.

[1] We use synonymously the terms "fractional Laplacian" and "Riesz fractional derivative".