Chaos, Solitons and Fractals 82 (2016) 87–96

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Invariance and computation of the extended fractal dimension for the attractor of CGL on $\mathbb{R}$

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ARTICLE INFO

Article history:
Received 5 May 2015
Accepted 30 October 2015
Available online 28 November 2015

MSC:
37B40
35Q56
35B41
37L30
35R15
37C45
28D20
47A35

Keywords:
Complexity
Entropy
Extended fractal dimension
Fractal dimension per unit length
Functional dimension
Global attractor

ABSTRACT

The main goal of this paper is to analyze the complexity of the asymptotic behavior of dissipative systems. More precisely, we want to explain how we can introduce the notion of extended fractal dimension in the case of infinite dimensional sets. In particular, we study the global attractor associated with the extended dynamical system induced by the complex Ginzburg–Landau equation on the line CGL. Furthermore, we compute and investigate the invariance of these quantities under an infinite type of metrics. As a direct consequence, we found that the attractor is similar in terms of complexity to an $L^\infty(\mathbb{R})$-ball in the space of band-limited functions.

1. Introduction

This article, which is at the interface between partial differential equations, ergodic theory and functional analysis, deals with the complexity of global attracting sets for dynamical systems provided by a parabolic PDE. Specifically, the complex Ginzburg–Landau equation CGL on the whole line $(x \in \mathbb{R})$, which is written as follows:

$$\partial_t u = (1 + i\alpha)\Delta u + u - (1 + i\beta)u|u|^2, \tag{1}$$

where the unknowns u map $\mathbb{R}_t^+ \times \mathbb{R}_x$ into $\mathbb{C}$ and α, β are parameters in $\mathbb{R}$. It is well-known, as presented in [1], that this equation on $\mathbb{R}$ possesses a global attractor denoted by $\mathcal{A}$, which attracts in $L^\infty_{\text{loc}}(\mathbb{R})$ all the trajectories. The complexity of such attractors has been studied by many researchers, for example: [2–9] and [10]. Most of those authors have pointed out that these types of attractors is infinite dimensional (see also Section 3.2). The complexity of such infinite dimensional attractors has been first studied by Chepyzhov and Vishik [11], by introducing the notion of Kolmogorov ε-entropy per unit length [12] in this framework. They focus mainly on uniform attractors in bounded domain. However, this approach seems to be particularly well suited for studying attractors on unbounded domains.

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http://dx.doi.org/10.1016/j.chaos.2015.10.037

A pioneering work on CGL is due to Collet and Eckmann [2]. They prove that ε-entropy per unit length in L^∞ framework is finite. Similarly, Zelik proves this result in several frameworks, namely using different metrics based on Sobolev spaces [8,9] and [10]. In [5], we introduced the rescaled L^2 and H^k norms, which seem to be adequate for computing ε-entropy per unit length. These norms could be used, at least, to obtain the subadditivity property (26) and, therefore, the existence of the limit of the ε-entropy per unit length (Formula (5) in Definition 3.3). Moreover, this rescaled norms allow introducing another way to define the topological entropy per unit length as presented in [6]. We want also to highlight that the existence of the limit in Formula (5) is more important in the case of topological entropy per unit length, because of the use of lim sup to define the dimension of type (6). Furthermore, we believe that this framework can be more suitable for a larger class of PDEs than those of parabolic type.

Furthermore, using other tools introduced by Kolmogorov and Tikhomirov in [12] for studying the notion of ε-entropy of compact sets in functional spaces, Zelik gave in [8,9] and [10] estimations of the orders of the ε-entropy growth for this global attractor using different frameworks.

In this article, we put more emphasis on the usefulness of using orders of growth of both ε-entropy per unit length and ε-entropy of such functional sets to go further and talk about *extended fractal dimension*. Here, in some sense, we mean by extended fractal dimension of a given infinite dimensional set an adequate numerical characteristic that is similar to a dimension. Indeed, as the well-known fractal dimension (4) in finite dimension, the extended fractal dimension is based on the ε-entropy or ε-entropy per unit length. In other words, extended fractal dimension is equivalent to fractal dimension but in infinite dimension. Examples of such notions are formulas (6)–(8) given below.

One of the main objectives of using the notion of extended fractal dimension is to be able to compute quantitative quantities in order to compare rigorously the size and the complexity of infinite dimensional sets.

In a sense, in this paper we deal with two extended fractal dimensions.The first one is *fractal dimension per unit length* introduced in [2], which is induced by the ε-entropy per unit length. The second one is *functional dimension* introduced in [12], based on the order of growth of the ε-entropy of the global attractor. More precisely, the main contributions of this article are:

- Theorem 3.2: we quantify the degree of chaos or freedom for this attractor using the functional dimension and investigate the invariance of this quantity under both $H^k_\rho(\mathbb{R})$ and $W^{k,\infty}_\rho(R)$ metrics ($\forall k \geq 0$). In this case, we are even able to compute exactly this quantity.
- Theorem 3.3: we show that the fractal dimension per unit length is an invariant for all $W^{k,\infty}(-L, L)$ and rescaled $H^k(-L, L)$ metrics. Indeed, in finite dimension, the fractal dimension is invariant because of the equivalence of norms, but in infinite dimension the situation becomes much more complicated.

In fact, this attractor contains bounded trajectories that are analytical functions in space. Thus, Collet and Eckmann in [2] prove, using ε-entropy per unit length, that this global attractor is much smaller in terms of complexity than the space of functions which are analytical and bounded in a strip. As a consequence of Theorem 3.3, we have established in this article that in fact this attractor is similar in terms of complexity to the space of analytical functions f whose Fourier transform $\widehat{f}$ are compactly supported.

The rest of this article is organized as follows: in Section 2 we highlight some well known facts about the dynamics of the solutions of CGL equation on the line. We also make precise the notion of (Z, Z_ρ)-attractor used in this paper. Throughout Section 3, we focus on how to introduce the notion of extended fractal dimension, mainly, by discussing the finite dimensional case to understand the infinite dimensional one, we then give our main result. Furthermore, in Section 4 we give the proofs of the two main theorems of this paper.

2. General framework for the CGL equation on the line

2.1. Notation and conventions

To begin with, we have to introduce some notations and conventions. Throughout this article we will use constants denoted by $c, c', C, K, \ldots$ that may vary from one line to one another, and that may depend on the data α, β of CGL.

For a given Hilbert space as $L^2(B)$ wherein B is an interval included in $\mathbb{R}$, the scalar product of two functions reads $\mathrm{Re}(\int_B u\bar{v}dx)$, where Re denote the real part.

We also use the standard Landau notations $f = O(g)$ if there exists a numerical constant C such that $f \leq Cg$. We set $f \simeq g$ if $f = O(g)$ and $g = O(f)$. We write $f = o(g)$, say for $L \to +\infty$, if for any $\varepsilon > 0$, then for $L \geq L_\varepsilon$, $f = O(\varepsilon g)$.

2.2. Functional spaces

Let us start this general framework by defining the functional spaces used. We introduce first a positive function $\rho : \mathbb{R} \to (0, \infty)$, which is called weight function with exponential growth, that is in $C_{loc}(\mathbb{R})$, bounded, and such that $\int_{\mathbb{R}} \rho(x)dx < +\infty$. We can also assume without loss of generality that $|\rho'(x)|, |\rho''(x)| \leq \rho(x)$ for all x. For instance $\rho(x) = e^{-|x|}$ works and we shall deal with this weight function in the sequel. Thus, we define the L^2_ρ space as follows

$$L^2_\rho = \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}) / \|u\|^2_{L^2_\rho} = \int_{\mathbb{R}} \rho|u|^2 dx < \infty \right\}.$$

And the L^∞ version,

$$L^\infty_\rho = \{ u \in L^\infty_{\mathrm{loc}}(\mathbb{R}) / \|u\|_{L^\infty_\rho} = \sup_{x\in\mathbb{R}} \{\rho(x)|u(x)|\} < \infty \}.$$

This functional spaces admit Sobolev versions for $k \geq 1$,

$$H^k_\rho = \left\{ u \in H^k_{\mathrm{loc}}(\mathbb{R}) / \|u\|^2_{H^k_\rho} = \int_{\mathbb{R}} \rho(x) \left(\sum_{i=0}^{k} |\partial_i u(x)|^2 \right) dx < \infty \right\},$$

$$W^{k,\infty}_\rho = \left\{ u \in W^{k,\infty}_{\mathrm{loc}}(\mathbb{R}) / \|u\|_{W^{k,\infty}_\rho} \right.$$

$$\left. = \sup_{x\in\mathbb{R}} \left\{ \rho(x) \left(\sum_{i=0}^{k} |\partial_i u(x)| \right) \right\} < \infty \right\}.$$