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New families of periodic orbits for a galactic potential

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ABSTRACT

We find analytically new families of periodic orbits of a Hamiltonian system which describes the local motion in the central area of a galaxy, whose dynamics have been studied by many authors.

1. Introduction and statement of the main results

In this paper we study the families of periodic orbits of a 3-dimensional (or simply 3D) isotropic harmonic oscillator perturbed by a polynomial potential

$$\mathcal{H} = \frac{1}{2}\left(P_1^2+P_2^2+P_3^2\right)+\frac{1}{2}\left(Q_1^2+Q_2^2+Q_3^2\right) + \varepsilon\mathcal{P}(Q_1,Q_2,Q_3), \tag{1}$$

where ε is a small parameter. The polynomial potential is

$$\mathcal{P}(Q_1,Q_2,Q_3) = Q_1^4+Q_2^4+Q_3^4 + a\left(Q_1^2Q_2^2+Q_1^2Q_3^2+Q_2^2Q_3^2\right), \tag{2}$$

where $a \in \mathbb{R}$ is a parameter.

The potential here studied

$$V = \frac{1}{2}\left(Q_1^2+Q_2^2+Q_3^2\right) + \varepsilon\left(Q_1^4+Q_2^4+Q_3^4+a\left(Q_1^2Q_2^2+Q_1^2Q_3^2+Q_2^2Q_3^2\right)\right)$$

is a 3-dimensional perturbed harmonic oscillator and describes the local motion in the central area of a galaxy. These local 2-or 3-dimensional potentials, become of the expansion of global galactic potentials in a Taylor series near a stable equilibrium point and have been extensively studied in order to describe the local motion inside the galaxies. This potential has been studied by many authors, see for instance Deprit and Elipe [4], Caranicolas [3], Elipe and Deprit [5], Elipe [6], Arribas et al. [2], Zotos [11–14], Zotos and Caranicolas [15], Zotos and Carpintero [16] and others.

In paper [10] the authors studied analytically the families of periodic orbits of the Hamiltonian (1) with (2) using the averaging theory, and they find several families of periodic orbits. Here we improve the results of [10] finding new families of periodic orbits, also using a result based in the averaging theory. The key point for obtaining these new families of

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periodic orbits is to work with the Lissajous variables instead of working directly with the Cartesian variables $(Q_1, Q_2, Q_3, P_1, P_2, P_3)$ as in [10]. The so-called Lissajous variables, name invented by Deprit in order to have the order zero 2D Hamiltonian and in 1:1 resonance as only one conjugate moment. Later on, the set of Lissajous variables was used and extended by Deprit and Elipe [4,5] for n oscillators and whatever resonance, and later on by Elipe [7].

The 3D Lissajous variables (L, l, G, g, N, n) are defined through the transformation

$$\mathcal{L} \,:\, (Q_1, Q_2, Q_3, P_1, P_2, P_3) \mapsto \;(L, G, N, l, g, n) : \mathbb{R}^6 \to \Omega \times \gamma$$

given by

$$Q_1 = \sqrt{G+N}\sin(l+g+n),\;\; P_1 = \sqrt{G+N}\cos(l+g+n),$$
$$Q_2 = \sqrt{L-G}\sin(l-g+n),\;\; P_2 = \sqrt{L-G}\cos(l-g+n),$$
$$Q_3 = \sqrt{L-N}\sin(l+g-n),\;\; P_3 = \sqrt{L-N}\cos(l+g-n),$$

where

$$\Omega = \{(L, G, N) \in \mathbb{R}^3 : L > 0, |G| < L, |N| < L\}$$

and γ is the torus

$$\{(l, g, n) \in \mathbb{R}^3/(l, g, n) \in [0, 2\pi)^3\}.$$

The 3D Lissajous transformation is a canonical transformation, i.e. the symplectic structure remains the standard one. In the new coordinates the Hamiltonian (1) becomes

$$\mathcal{H} = L + \varepsilon \mathcal{P}_1(L, l, G, g, N, n). \qquad (3)$$

where $\mathcal{P}_1(L, l, G, g, N, n)$ is the pullback of the 3D Lissajous transformation with the perturbed polynomial $\mathcal{P}$.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small in the invariant set $\mathcal{H} = h > 0$, the Hamiltonian system defined by the Hamiltonian (3) with the perturbation given by (2) has the following new families of $2\pi-$periodic solutions in the variable l.*

If $a \in (-6, 6) \setminus \{0, 2\}$ we have the families

(I): $\gamma_\varepsilon^I(l) = (L(l,\varepsilon), G(l,\varepsilon), g(l,\varepsilon), N(l,\varepsilon), n(l,\varepsilon))$ *such that*

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^I(0) = \left(h, \frac{(a+6)h}{18-a}, \frac{k\pi}{2}, \frac{3(2-a)h}{18-a}, \frac{\pi}{4} + \frac{m\pi}{2}\right)$$

for $k, m = 0, 1, 2, 3$.

(II): $\gamma_\varepsilon^{II}(l) = (L(l,\varepsilon), G(l,\varepsilon), g(l,\varepsilon), N(l,\varepsilon), n(l,\varepsilon))$ *such that*

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^{II}(0) = \left(h, \frac{3(2-a)h}{18-a}, \frac{\pi}{4} + \frac{k\pi}{2}, \frac{(a+6)h}{18-a}, \frac{m\pi}{2}\right)$$

for $k, m = 0, 1, 2, 3$.

(III): $\gamma_\varepsilon^{III}(l) = (L(l,\varepsilon), G(l,\varepsilon), g(l,\varepsilon), N(l,\varepsilon), n(l,\varepsilon))$ *such that*

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^{III}(0) = \left(h, \frac{(a+6)h}{18-a}, \frac{\pi}{4} + \frac{k\pi}{2}, \frac{(a+6)h}{18-a}, \frac{\pi}{4} + \frac{m\pi}{2}\right)$$

for $k, m = 0, 1, 2, 3$.

If $a \in \mathbb{R} \setminus \{0, 2\}$ we have the families

(IV): $\gamma_\varepsilon^I(l) = (L(l,\varepsilon), G(l,\varepsilon), g(l,\varepsilon), N(l,\varepsilon), n(l,\varepsilon))$ *such that*

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^I(0) = \left(h, \frac{h}{3}, \frac{k\pi}{2}, \frac{h}{3}, \frac{m\pi}{2}\right)$$

for $k, m = 0, 1, 2, 3$.

Table 1
The orbits of Theorem 1 in Cartesian coordinates where $\Gamma = \sqrt{\frac{2h}{3}}$, $\Gamma_1 = \sqrt{\frac{2(a-6)h}{a-18}}$ and $\Gamma_2 = \sqrt{\frac{2(a+6)h}{18-a}}$.

Families	I	$Q_1(t,\varepsilon)$	$-\Gamma_1 \sin\left(t-\frac{\pi}{4}\right) + O(\varepsilon)$
			$\Gamma_1 \sin\left(t+\frac{\pi}{4}\right) + O(\varepsilon)$
		$Q_2(t,\varepsilon)$	$\mp\Gamma_1 \sin\left(t-\frac{\pi}{4}\right) + O(\varepsilon)$
			$\mp\Gamma_1 \sin\left(t+\frac{\pi}{4}\right) + O(\varepsilon)$
		$Q_3(t,\varepsilon)$	$\mp\Gamma_2 \sin\left(t+\frac{\pi}{4}\right) + O(\varepsilon)$
			$\mp\Gamma_2 \sin\left(t-\frac{\pi}{4}\right) + O(\varepsilon)$
	II	$Q_1(t,\varepsilon)$	$-\Gamma_1 \sin\left(t-\frac{\pi}{4}\right) + O(\varepsilon)$
			$-\Gamma_1 \sin\left(t+\frac{\pi}{4}\right) + O(\varepsilon)$
		$Q_2(t,\varepsilon)$	$\mp\Gamma_2 \sin\left(t+\frac{\pi}{4}\right) + O(\varepsilon)$
			$\mp\Gamma_2 \sin\left(t-\frac{\pi}{4}\right) + O(\varepsilon)$
		$Q_3(t,\varepsilon)$	$\mp\Gamma_1 \sin\left(t-\frac{\pi}{4}\right) + O(\varepsilon)$
			$\mp\Gamma_1 \sin\left(t+\frac{\pi}{4}\right) + O(\varepsilon)$
	III	$Q_1(t,\varepsilon)$	$-\Gamma_2 \cos t + O(\varepsilon)$
			$-\Gamma_2 \sin t + O(\varepsilon)$
		$Q_2(t,\varepsilon)$	$\mp\Gamma_1 \sin t + O(\varepsilon)$
			$\mp\Gamma_1 \cos t + O(\varepsilon)$
		$Q_3(t,\varepsilon)$	$\mp\Gamma_1 \sin t + O(\varepsilon)$
			$\mp\Gamma_1 \cos t + O(\varepsilon)$
	IV	$Q_1(t,\varepsilon)$	$-\Gamma \cos t + O(\varepsilon)$
			$-\Gamma \sin t + O(\varepsilon)$
		$Q_2(t,\varepsilon)$	$\pm\Gamma \cos t + O(\varepsilon)$
			$\mp\Gamma \sin t + O(\varepsilon)$
		$Q_3(t,\varepsilon)$	$\pm\Gamma \cos t + O(\varepsilon)$
			$\mp\Gamma \sin t + O(\varepsilon)$

Theorem 1 is proved in Section 2.

If we write the periodic orbits described in Theorem 1 in Lissajous coordinates (L, l, G, g, N, n) in Cartesian coordinates $(Q_1, Q_2, Q_3, P_1, P_2, P_3)$ we obtain Table 1.

From Table 1 is easy to obtain the implicit equations of two of the periodic orbits of the family I for $\varepsilon = 0$, which are given by the intersection of the elliptic cylinder $\frac{Q_1^2}{\Gamma_1^2} + \frac{Q_3^2}{\Gamma_2^2} = 1$ with the planes $Q_1 = \pm Q_2$. Similarly for the other periodic orbits of the family I.

Again from Table 1 it follows that the family II comes from the intersection of the elliptic cylinder $\frac{Q_1^2}{\Gamma_1^2} + \frac{Q_2^2}{\Gamma_2^2} = 1$ with the planes $Q_1 = \pm Q_3$. Similarly for the other periodic orbits of the family I.

The implicit equations of two periodic orbits of the family III are given by the intersection of the elliptic cylinder $\frac{Q_1^2}{\Gamma_2^2} + \frac{Q_2^2}{\Gamma_1^2} = 1$ with the elliptic cylinder $\frac{Q_1^2}{\Gamma_2^2} + \frac{Q_3^2}{\Gamma_1^2} = 1$. Similarly for the other periodic orbits of this family.

The implicit equation for the orbits of the family IV are $Q_1 = \pm Q_2 = \pm Q_3$.

We must mention that in the paper [10] three more additional families of periodic orbits of the Hamiltonian system