Chaos, Solitons and Fractals 78 (2015) 33–38

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Periodic solutions in reaction–diffusion equations with time delay

Li Li[a,b,*]

[a] Key Laboratory of Computational Intelligence and Chinese Information, Processing of Ministry of Education, Shanxi University, Taiyuan 030006, People's Republic of China
[b] School of Computer and Information Technology, Shanxi University, Taiyuan 030006, People's Republic of China

ARTICLE INFO

Article history:
Received 11 May 2015
Accepted 3 July 2015
Available online 25 July 2015

PACS:
87.23.Cc
82.40.Bj
05.45.Pq

Keywords:
Diffusion system
Time delay
Periodic solution
Fredholm mapping
Coincidence degree

ABSTRACT

Spatial diffusion and time delay are two main factors in biological and chemical systems. However, the combined effects of them on diffusion systems are not well studied. As a result, we investigate a nonlinear diffusion system with delay and obtain the existence of the periodic solutions using coincidence degree theory. Moreover, two numerical examples confirm our theoretical results. The obtained results can also be applied in other related fields.

1. Introduction

Consider the diffusion system of the form

$$\frac{\partial u(r,t)}{\partial t} = D\Delta u(r,t) + f(u(r,t)), t \geq 0, r \in \Omega \subset R^m, \quad (1)$$

where $u \in R^n$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $d_i > 0$, $i = 1, 2, \ldots, n$, and Δ is the Laplace operator, that is,

$$\Delta u(r,t) = \left(\sum_{k=1}^{m} \frac{\partial^2 u_1(r,t)}{\partial r_k^2}, \ldots, \sum_{k=1}^{m} \frac{\partial^2 u_n(r,t)}{\partial r_k^2} \right)^T.$$

Let $\theta = (\theta_1, \theta_2, \cdots, \theta_m)$ be a united vector and c be a constant, then $u(r,t) = \varphi(r \cdot \theta + ct)$ is called as a traveling wave solution of (1). Thus, we have the ordinary differential system

$$D\varphi''(\xi) - c\varphi'(\xi) + f(\varphi(\xi)) = 0. \quad (2)$$

The existence and stability of traveling wave solutions for system (1) have been extensively studied, see more details in Refs. [1–10]. In [11], Schaaf first systematically studied two scalar reaction–diffusion equations with a single discrete delay. Recently, the existence of traveling wave solutions for delay diffusion system has attracted considerable attention [12–18]. However, most papers only considered the existence of traveling wavefronts [19–21].

Now we consider the diffusion system of the following form:

$$\begin{cases} \dfrac{\partial u(r,t)}{\partial t} = d_1 \sum\limits_{k=1}^{m} \dfrac{\partial^2 u(r,t)}{\partial r_k^2} \\ \quad + f_1(u(r,t), v(r,t), u(r,t-l_1), v(r,t-l_2)), \\ \dfrac{\partial v(r,t)}{\partial t} = d_2 \sum\limits_{k=1}^{m} \dfrac{\partial^2 v(r,t)}{\partial r_k^2} \\ \quad + f_2(u(r,t), v(r,t), u(r,t-l_3), v(r,t-l_4)), \end{cases} \quad (3)$$

where $t \geq 0$, $r \in \Omega \subset R^m$, $d_1 > 0$, $d_2 > 0$ are the diffusion coefficients. Let $u(r,t) = \varphi(r \cdot \theta + c_1 t)$ and $v(r,t) = \phi(r \cdot \theta + c_2 t)$.

---

[*] Tel.: +86 13513649570.
E-mail address: lili831113@163.com

http://dx.doi.org/10.1016/j.chaos.2015.07.002
0960-0779/

We have that:

$$\begin{cases} d_1\varphi''(\xi)-c_1\varphi'(\xi) \\ \quad +f_1(\varphi(\xi),\phi(\xi),\varphi(\xi-c_1l_1),\phi(\xi-c_2l_2))=0, \\ d_2\phi''(\xi)-c_2\phi'(\xi) \\ \quad +f_2(\varphi(\xi),\phi(\xi),\varphi(\xi-c_1l_3),\phi(\xi-c_2l_4))=0. \end{cases} \tag{4}$$

When the diffusion coefficients d_1 and d_2 and delays l_1, l_2, l_3 and l_4 are T-periodic functions, Eq. (4) can be written as:

$$\begin{cases} x''(t)+a(t)x'(t) \\ \quad -f_1(t,x(t),y(t),x(t-\tau_1(t)),y(t-\sigma_1(t)))=0, \\ y''(t)+b(t)y'(t) \\ \quad -f_2(t,x(t),y(t),x(t-\tau_2(t)),y(t-\sigma_2(t)))=0, \end{cases} \tag{5}$$

where a, b, τ_1, τ_2, σ_1, σ_2, f_1 and f_2 are T-periodic functions. In this case, if (5) has a T-periodic solution $(x(t), y(t))$, we have

$$\begin{cases} x'(t)=\int_t^{t+T}\frac{\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-1} \\ \quad \times f_1(s,x(s),y(s),x(s-\tau_1(s)),y(s-\sigma_1(s)))ds, \\ y'(t)=\int_t^{t+T}\frac{\exp\left(\int_t^s b(u)du\right)}{\exp(\int_0^T b(u)du)-1} \\ \quad \times f_2(s,x(s),y(s),x(s-\tau_2(s)),y(s-\sigma_2(s)))ds. \end{cases} \tag{6}$$

In fact, system (3) has a traveling wave solution if, and only if system (6) has a T-periodic solution. As a result, the purpose of this paper is to establish the condition for the existence of at least one T-periodic solution of system (6), by using continuation theorem [22].

2. Some preparation

In this section, we will give some preparations which are crucial in the proof of our theorem. For the sake of discussion, in what follows we will introduce this theorem as follows.

Let X and Y be two real Banach spaces, L: $\mathrm{domL}\subset \mathrm{X}\to \mathrm{Y}$ be a Fredholm mapping of index zero, and $P:X\to X$, $Q:Y\to Y$, be continuous projections such that $\mathrm{Im\,P}=\mathrm{KerL}$, $\mathrm{KerQ}=\mathrm{Im\,L}$, and $X=\mathrm{KerL}\oplus\mathrm{KerP}$, $Y=\mathrm{Im\,L}\oplus\mathrm{Im\,Q}$. Denote the restriction of L to $\mathrm{domL}\cap\mathrm{KerP}$ as by L_p, $K_p:\mathrm{Im\,L}\to\mathrm{domL}\cap\mathrm{KerP}$ as the inverse of L_p, and an isomorphism of ImQ onto KerL by $J:\mathrm{ImQ}\to\mathrm{KerL}$.

Lemma 1 ([22]). *Let $\Omega\subset X$ be an open bounded set and let $N:X\to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$ (i.e., $QN:\overline{\Omega}\to Y$ and $K_p(I-Q)N:\overline{\Omega}\to X$ are compact). Assume*

- (a) *$Lx\neq\lambda Nx$ for every $(x,\lambda)\in(\mathrm{domL}\backslash\mathrm{Ker}\cap\partial\Omega)\times(0,1)$;*
- (b) *$Nx\notin\mathrm{ImL}$ for every $x\in\mathrm{KerL}\cap\partial\Omega$;*
- (c) *$\deg(QN|_{\ker L},\Omega\cap\ker L,0)\neq 0$.*

Then the equation $Lx=Nx$ has at least one solution in $\mathrm{domL}\cap\overline{\Omega}$.

Throughout this paper, we will discuss the problem in several classical spaces $C(R,\ R)$, $C^1(R,\ R)$. For the $x\in C(R,\ R)$, where $x=(x_1,x_2)^T$, we use the norm $\|x_i\|_\infty=\max_{t\in[0,T]}\left|x_i(t)\right|$ $(i=1,2)$ and $\|x\|_\infty=\max\{\|x_1\|_\infty,\|x_2\|_\infty\}$. Moreover, we will adopt the notation $|x|_k=(\int_0^T\left|x(t)\right|^k dt)^{1/k}$.

Banach space $X=\{x|\ x\in C(R,R), x(t)=x(t+T),$ for all $t\in R\}$ has the norm $\|x\|_X=\|x\|_\infty$, and Y is also a real Banach space.

Now we can define L as the linear operator from $\mathrm{domL}\subset\mathrm{X}$ to Y with

$$\mathrm{domL}=\{\mathrm{x}|\ \mathrm{x}\in\mathrm{X},\mathrm{x}'\in\mathrm{C(R,R)}\ \text{and}\ \mathrm{x}(0)=0\}$$

and

$$L(x)=(x'),\qquad x=(x_1,x_2)^T\in\mathrm{domL}.$$

Define the nonlinear operator N: $X\to Y$ by

$$N(x)=\int_t^{t+T}G(t,s)f(s,x_1(s),x_2(s),x_1(s-\tau(s)), \\ x_2(s-\sigma(s)))ds,$$

where $G(t,s)=\left(\begin{smallmatrix}G_1(t,s)\\ G_2(t,s),\end{smallmatrix}\right)$ with

$$G_1(t,s)=\frac{\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-1},$$

$$G_2(t,s)=\frac{\exp\left(\int_t^s b(u)du\right)}{\exp\left(\int_0^T b(u)du\right)-1},$$

and $f=\left(\begin{smallmatrix}f_1\\ f_2,\end{smallmatrix}\right)$ $\tau=\left(\begin{smallmatrix}\tau_1\\ \tau_2,\end{smallmatrix}\right)$ $\sigma=\left(\begin{smallmatrix}\sigma_1\\ \sigma_2\end{smallmatrix}\right)$ and $D=\left(\begin{smallmatrix}D_1\\ D_2.\end{smallmatrix}\right)$

It is clear to see that there exists a vector constant $M>0$ $(M\in R^2)$ such that for any $t\in R$,

$$G(t,s)\le M.$$

Then, we can consider the operator equation

$$L(x)=\lambda N(x). \tag{7}$$

It is trivial to see that L is a bounded linear operator with

$$\mathrm{KerL}=\{x\in\mathrm{domL}:\mathrm{x(t)=d,t\in R,d\in R^2}\},$$

$$\mathrm{Im\,L}=\left\{y\in Y:\int_t^{t+T}y(s)ds=0\right\},$$

and

$$\dim\mathrm{KerL}=2=\mathrm{co}\dim\mathrm{Im\,L}.$$

Consequently, it follows that L is a Fredholm mapping of index zero.

Define P: $X\to X$ and Q: $Y\to Y$ respectively as

$$Px=x(0),\quad x\in X,$$

and

$$Qy=\frac{1}{T}\int_0^T y(s)ds,\quad y\in Y.$$

It is not difficult to show that P and Q are continuous projectors such that

$$\mathrm{Im\,P}=\mathrm{KerL},\quad \mathrm{Im\,L}=\mathrm{KerQ}=\mathrm{Im\,}(I-Q).$$

Furthermore, the generalized inverse (to L) K_P: $\mathrm{Im\,L}\to\mathrm{domL}\cap\mathrm{KerP}$ exists and has the following form:

$$K_p(y)=\int_0^t y(s)ds.$$

In fact, for $y\in\mathrm{ImL}$, we have

$$(LK_p)y(t)=[(K_py)(t)]'=y(t).$$