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New aspects of symmetry of elementary cellular automata

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ABSTRACT

We present a new classification of elementary cellular automata. It is based on the structure of the network of states, connected with the transitions between them; the latter are determined by the automaton rule. Recently an algorithm has been proposed to compress the network of states (Krawczyk, Physica A 390 (2011) 2181). In this algorithm, states are grouped into classes, according to the local symmetry of the network. In the new classification, an automaton is described by the number of classes #(N) as dependent on the system size N. In most cases, the results reflect the known classification into 88 groups. However, the function #(N) also appears to be the same for some rules which have not been grouped together yet. In this way, the automaton 23 is equivalent to 232, 77 to 178, 105 to 150, the pair (43, 113) to the pair (142, 212) and the group (12, 68, 207, 221) to the group (34, 48, 187, 243). Furthermore, automata 51, 204, the pair (15, 85) and the pair (170, 240) are all mutually equivalent. Results are also presented on the structure of networks of states.

1. Introduction

Attempts of cellular automata classification were made by several authors. Proposed approaches are based, either on observations of the automata behaviour, or properties of the rules themselves. The most famous classification of the elementary cellular automata CA belongs to the first category, and was introduced by Wolfram [1]. The classification regarded the character of automata evolution and led to the division of the whole set of elementary rules into four classes. The first group includes rules which when applied to a given sequence cause removal of any randomness, and as a consequence lead to a stable state. The rules included into the second class lead to a stable or oscillating character of evolution. The third class covers rules which show with pseudo-random or chaotic behaviour. The fourth class is especially interesting as it covers rules which lead to the complex evolution. The classification makes it possible to see differences in behaviour of the system observed during the evolution ruled by the particular automata.

Besides the Wolfram's classification there are also different propositions. One of them was introduced by Li et al. [2]. The classification is based on the observation of the asymptotic behaviour of the elementary CA, and, as a result, six types are distinguished. CA for which evolution leads to: (a) spatially homogeneous fixed point, (b) spatially inhomogeneous fixed point or a uniform shift of a fixed pattern, (c) periodic behaviour or shifted periodic behaviour, (d) locally chaotic behaviour, (e) chaotic behaviour and (f) complex behaviour. Another classification is based on the concept of equicontinuity which originally was introduced by Gilman [3] and then extended by Kurka [4], where an equicontinuity set contains a set of its Lyapunov stable points. In this scheme, similarly to Wolfram's classification, each CA can be included into one of the four categories: equicontinuous CA, CA with some equicontinuous points, sensitive but not positively expansive and positively expansive. It is not clear a priori to which category a particular automaton belongs, but for the elementary CA this problem can be solved [5]. A review of the classification methods, along with a discussion of problems that arise when attempting classification of

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Table 1
Rule equivalence. The bar means negation, i.e. $\bar{0} = 1$ and $\bar{1} = 0$.

Rule	A	B	C	D	E	F	G	H
Conjugation	$\bar{H}$	$\bar{G}$	$\bar{F}$	$\bar{E}$	$\bar{D}$	$\bar{C}$	$\bar{B}$	$\bar{A}$
Reflection	A	E	C	G	B	F	D	H
Conjugation+reflection	$\bar{H}$	$\bar{D}$	$\bar{F}$	$\bar{B}$	$\bar{G}$	$\bar{C}$	$\bar{E}$	$\bar{A}$

automata, is presented by Sutner in [6]. Analysis of cellular automata properties is an important problem as they have a lot of applications in modelling of physical processes, such as diffusion, phase transition, wave propagations or road traffic analysis [7].

In this paper we propose a method of class identification which is based on the similarities observed in patterns of connections between possible states of the analysed system. Here we say that two states are connected if one state is obtained from another one, according to a given automaton rule. The presented method takes into account the symmetry observed in the state space, and it allows us to indicate equivalent – according to symmetry – rules. The method is exhaustive, and is based on scrupulous analysis performed for sequences of lengths up to $N = 19$ (in some cases longer if the character of results does not allow conclusions to be drawn earlier).

The paper is organised as follows. In Section 2 the analysed system is presented. In Section 3 the procedure for identification of classes is described. Section 4 is devoted to discussion about the obtained results, and Section 5 provides their summary.

2. Analysed system and symmetry of rules

We analyse a one-dimensional cellular automaton, with two possible states of each cell, 0 and 1. The rule of the change of a state of a cell is determined on the basis of the state of the cell itself and its two nearest neighbours, with usage of periodic boundary conditions. Namely, the rule is a chain $(A, B, \ldots, H)$ of outcomes of the states of these three cells, in a decreasing order $(111, 110, \ldots, 000)$. Such a definition leads to $2^8 = 256$ different rules of the evolution process, so called elementary cellular automata. It was however shown that the number of unique rules is lower, as some of the rules are equivalent [8]. The rules which are equivalent to a given one are obtained as its conjugation, reflection and combined operation of conjugation and reflection (see Table 1). As a result, the set of all 256 elementary automata is reduced into 88 unique rules, or in other words, to 88 groups of equivalent rules. The same result was recently obtained in [9] by an analysis of the symmetry of the rules. It may happen, however, that after any change, the obtained rule is the same as at the beginning. Because of that, symmetry groups do not always contain four rules. To be precise, there are 44 groups which contain 4 rules each, 36 groups which contain 2 rules each, and a remaining 8 rules which form one-element groups. Illustrative examples are presented in Table 2.

Consider a system consisting of all possible sequences of zeros and ones of a specified length N. Each sequence can be converted to an exact single sequence determined by the automaton rule which is currently being considered. Each sequence may, however, be obtained by the transformation of

Table 2
Examples of groups of equivalent rules: (a) group contains 4 different rules No 7, 21, 31 and 87, (b) group contains 2 different rules No 132 and 222, (c) group contains rule No 77.

(a)	*Rule*	0	0	0	0	0	1	1	1	*No 7*
	Conjugation	0	0	0	1	1	1	1	1	*No 31*
	Reflection	0	0	0	1	0	1	0	1	*No 21*
	Conjugation+reflection	0	1	0	1	0	1	1	1	*No 87*
(b)	*Rule*	1	0	0	0	0	1	0	0	*No 132*
	Conjugation	1	1	0	1	1	1	1	0	*No 222*
	Reflection	1	0	0	0	0	1	0	0	*No 132*
	Conjugation+reflection	1	1	0	1	1	1	1	0	*No 222*
(c)	*Rule*	0	1	0	0	1	1	0	1	*No 77*
	Conjugation	0	1	0	0	1	1	0	1	*No 77*
	Reflection	0	1	0	0	1	1	0	1	*No 77*
	Conjugation+reflection	0	1	0	0	1	1	0	1	*No 77*

different numbers of sequences. The exact number of states (sequences) leading to a given state depends on the automaton rule and the system size N. The state space considered can be represented as a directed graph, where 2^N nodes are identified with the possible states of the system. As for a given automaton, each state can be transformed to only one other state the out-degree of all nodes is equal to 1. However, the in-degree is different for different nodes.

3. Classes

Let us consider the given automaton rule. Analysis of the pattern of ties in the obtained graph allows us to indicate nodes which are similar, which means that the structure of connections of some nodes is the same. An example of the obtained graph – for CA No 86 and $N = 4$ is presented in Fig. 1. Because of that, nodes can be divided into classes, and each class contains nodes which are similar. As a result, the state space formed by all possible classes is smaller than the initial network of states. The rate of the reduction depends on the system size.

The concept of the reduction of the system size is based on the observation that, often in real systems, some kind of symmetry can be uncovered. If it is the case, it is possible to specify groups of states of the analysed system which manifest similar properties. When one thinks about the system like a graph, the mentioned similarities concern primarily nodes degrees. However, this quantity by itself does not reflect the node properties. If two nodes have the same degree, but the degrees of their neighbours are different, the nodes cannot be classified as being similar. If weights of particular links in a graph are not equal, which is not the case here, one must also distinguish identical patterns of connections but with different weights.

In the case of cellular automata the graph of transitions between states is directed. Therefore, the class of states is determined both by the list of states which for a particular automaton lead to a given state, and the one-element list, which preserve information about the state which is obtained from the one actually being considered. An example of the class identification procedure is presented in Table 3 for the automaton No 86 and $N = 4$. The parameter which differentiate two nodes is their degree, so, at the very beginning of our procedure, all nodes which have the same degree are labelled with the same symbol. With this done, the next step is to