Chaos, Solitons and Fractals 78 (2015) 107–117

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Limit cycle bifurcations near homoclinic and heteroclinic loops via stability-changing of a homoclinic loop[☆]

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ARTICLE INFO

Article history:
Received 23 August 2014
Accepted 14 July 2015
Available online 3 August 2015

MSC:
34C05
34C29
34C25

Keywords:
Piecewise smooth system
Stability
Homoclinic loop
Heteroclinic loop
Alien limit cycle

ABSTRACT

This paper is concerned with the problem of limit cycle bifurcation near a homoclinic or heteroclinic loop for non-smooth systems. By establishing the Poincaré map, some stability criteria are derived for a homoclinic loop in the non-smooth system under study. Furthermore, based on the theory of stability-changing of a homoclinic loop, a new approach is proposed to find limit cycles for the non-smooth system. Finally, several examples are provided to illustrate the obtained results.

1. Introduction

Recently, theoretical research and practical applications of piecewise smooth (PS for short) systems have attracted many researchers' attentions from a variety of disciplines including control theory, electronics, physics, medicine as well as biology, see [2,3,9] and references therein. PS systems can exhibit some complicated phenomena [3,9,21], such as sliding homoclinic bifurcation and sliding-crossing bifurcation, while these are forbidden in smooth systems.

As for PS systems, lots of research is focused on the theory of limit cycle bifurcations [1,4,8,13,15–17,23]. The authors in [4,13] studied the problem of Hopf bifurcation for non-smooth planar systems and it is found in [13] that two limit cycles can appear near a focus of either FF, FP, or PP type (see Theorem 3.3 of [13]) for piecewise linear systems. Furthermore, via the direct computation, it was shown that piecewise linear systems with two real saddles can have two limit cycles [1], and this conclusion will be confirmed by another method established in this paper, see Example 5.2. Some investigations about the number of limit cycles on piecewise linear systems can be seen in [15,16] and references therein. Recently, similar to the smooth case [18], by studying a perturbed piecewise Hamiltonian system, the authors in [17] have derived a formula for the generalized Melnikov function which can be used to study the number of limit cycles bifurcated from periodic orbits. Based on this method, local and global bifurcations for a class of PS systems were discussed in [23].

For a smooth planar system

$$\dot{x} = f_0(x, y), \quad \dot{y} = g_0(x, y), \tag{1.1}$$

some criteria for the stability of a homoclinic loop of it can be found in Theorem 3.3 in [5], Theorems 4.2.2 and 4.2.3 in [10], or Theorem 1.1 in [11], which can be stated as follows.

☆ The project was supported by National Natural Science Foundation of China (11271261 and 11431008) and the 7th European Community Framework Programme (FP7-PEOPLE-2012-IRSES-316338).

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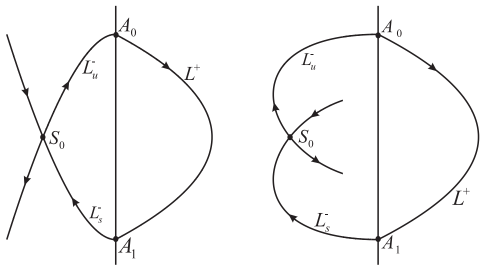
http://dx.doi.org/10.1016/j.chaos.2015.07.015
0960-0779/



(1) Convex case (2) Concave case

Fig. 1. Two possible cases of the homoclinic loop L.

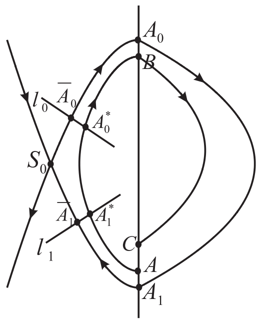


Fig. 2. Poincaré mape for the convex case near L.

Theorem 1.1. *Suppose that system (1.1) has an oriented clockwise homoclinic loop L_0 passing through a hyperbolic saddle S. Let*

$$\mu_1 = (f_{0x} + g_{0y})(S), \quad \mu_2 = \oint_{L_0} (f_{0x} + g_{0y})dt, \quad \mu_3 = R_1(S),$$

where $R_1(S)$ denotes the first saddle quantity of S. Then

(i) *The loop L_0 is orbitally stable (unstable) as $\mu_1 < 0(>0)$.*
(ii) *If $\mu_1 = 0$, then L_0 is orbitally stable (unstable) as $\mu_2 < 0(>0)$.*
(iii) *Let $\mu_1 = 0, \mu_2 = 0, \mu_3 \neq 0$. Then L_0 is orbitally stable if (a) $\mu_3 > 0$ and L_0 is convex, or (b) $\mu_3 < 0$ and L_0 is concave; otherwise, L_0 is unstable.*

We remark that in Theorem 1.1, if

$$\left.\frac{\partial(f_0, g_0)}{\partial(x, y)}\right|_{(x,y)=S} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \lambda > 0,$$

then

$$R_1(S) = \frac{1}{2\lambda}\left[f_{0xxy} + g_{0xyy} - \frac{1}{\lambda}(f_{0xx}f_{0xy} - g_{0xy}g_{0yy})\right]\Bigg|_{(x,y)=S}.$$

As can be seen from [10,12,22], μ_i defined in Theorem 1.1 can be used to study limit cycle bifurcations of the perturbed system of (1.1) near L_0. Meanwhile, some sufficient conditions presented in section 2 of Chapter 4 in [10] can also be utilized to obtain one, two or three limit cycles. Inspired by the above discussion, we aim to investigate the stability of a homoclinic and research the problem of limit cycle bifurcation near a homoclinic or heteroclinic loop (see Fig. 1 and 5) for PS systems.

The remainder of this paper is organized as follows. In Section 2, we mainly consider the stability of a homoclinic loop in a PS system and provide some criteria for it (see Theorem 2.1). In Section 3, we study the limit cycle bifurcations near a homoclinic loop for PS systems; we present a sufficient condition for a non-smooth system to have a homoclinic loop (see Theorem 3.1) and to have 2 or 3 limit cycles near a homoclinic loop (see Theorem 3.2). In Section 4, we investigate the bifurcation of limit cycles near a heteroclinic loop, and give the corresponding sufficient conditions with k ($k = 1, 2, 3$, or 4) limit cycles (see Theorem 4.1). In Section 5, two concrete examples are studied. Especially, it is proved that piecewise linear systems with two real saddles can have two limit cycles near a heteroclinic loop.

2. Stability of a homoclinic loop

In this section, we mainly study the stability of a homoclinic loop in a PS system by establishing a Poincaré map near the loop.

Consider a piecewise system of the form

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \tag{2.1}$$

where

$$f(x, y) = \begin{cases} f^+(x, y), & x \geq 0, \\ f^-(x, y), & x < 0, \end{cases} \quad g(x, y) = \begin{cases} g^+(x, y), & x \geq 0, \\ g^-(x, y), & x < 0, \end{cases} \tag{2.2}$$

with $f^{\pm}, g^{\pm} \in C^{\infty}$. Then, these functions define the following two C^{∞} systems

$$\begin{cases} \dot{x} = f^+(x, y), \\ \dot{y} = g^+(x, y) \end{cases} \tag{2.1a}$$

and

$$\begin{cases} \dot{x} = f^-(x, y), \\ \dot{y} = g^-(x, y), \end{cases} \tag{2.1b}$$

which are called right and left subsystems of (2.1), respectively, see [13].

Suppose that system (2.1) has a homoclinic loop $L = L^+ \cup L_s^- \cup L_u^- \cup S_0$ with a clockwise orientation, which intersects the y-axis at points A_0 and A_1 successively, where $S_0 \in \{(x, y)|x < 0\}$ is a hyperbolic saddle and

$$L^+ = \widehat{A_0A_1}, \quad L_s^- = \widehat{A_1S_0}, \quad L_u^- = \widehat{S_0A_0}.$$

Then, one can find two possible cases as shown in Fig. 1.

Let ρ be a sufficiently small positive number. Denoted by A is a point on the y-axis satisfying $A = A_1 - an_0$, where $n_0 = (0, 1)^T$ and $0 < -a < \rho$ in the convex case or $0 < a < \rho$ in the concave case. Then, the positive orbit, denoted by γ_A^+, of (2.1) starting at A intersects the y-axis at points $B = A_0 + a_3n_0$ and $C = A_1 - \tilde{P}(a_3)n_0$ in turn, where $a_3 = \bar{P}(a)$ and A, B, C are consecutive intersections of γ_A^+ with the y-axis, see Fig. 2. Define $\bar{P}(0) = \lim_{a\to 0}\bar{P}(a)$. Then, the orbit from A to C defines a map called the Poincaré map or the return map of system (2.1) near L. The map, denoted by P, satisfies

$$P(a) = \tilde{P}(a_3) = (\tilde{P} \circ \bar{P})(a), \tag{2.3}$$

where $-\rho < a \leq 0$ (in the convex case) or $0 \leq a < \rho$ (in the concave case), with $P(0) = 0$. Therefore, using the construction of $P(a)$, we introduce the following definition.