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Analytical expression for zero Lyapunov exponent of chaotic noised oscillators

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ABSTRACT

This paper is devoted to the analytical formula for zero Lyapunov exponent describing the dynamics of interacting chaotic systems with noise. The deduced analytical prediction is in a good agreement with the value of zero Lyapunov exponent obtained numerically for two unidirectionally coupled Rössler oscillators. We have shown that this good agreement is observed for a wide diapason of the values of the control parameters.

1. Introduction

There are no doubts that Lyapunov exponents (LEs) are very powerful tool used frequently to study the complex dynamics of nonlinear systems. The theory and procedures of Lyapunov exponent calculation have been properly developed [1–4] and now Lyapunov exponents are used widely in very different fields of science, including (but not limited to) physics [5], astronomy [6], medicine [7], economy [8], etc. Due to their great efficiency Lyapunov exponents are applied to a large number of complex systems, including spatially extended ones [9–15]).

Nowadays, among techniques devoted to the Lyapunov exponent calculation for nonlinear systems two main approaches may be distinguished generally, namely, (i) the estimation of the largest Lyapunov exponents from time series [4] of the examined system and (ii) the application of the standard procedures [3,4] based on the numerical calculation of Lyapunov sums with the help of the system evolution operator and its linearization. Estimation of the largest Lyapunov exponent from time series is very important for experiments (when the evolution operator is unknown) and used widely for the experimental data, including data of living systems (see, e.g., [16]). The second approach allows to calculate a spectrum of Lyapunov exponents but it requires an explicit form of the evolution operator of the system under study.

The very interesting, important and promising point is the analytical estimation of Lyapunov exponents. The analytical formula for the value of Lyapunov exponent can be rather easily obtained only for the steady-state solutions, namely, for the fixed points of nonlinear systems with a small number of degrees of freedom and for the steady-state spatially homogeneous solutions of the spatially extended systems. More interestingly, the value of Lyapunov exponent has been

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obtained analytically for simple models of nonlinear oscillators being under noise [17–20] (typically, in connection with the problem of synchronization of ensemble of oscillators by common external noise) as well as for the neural oscillator models [21–23].

The next step in the use of the Lyapunov exponent apparatus is the problem concerning the analytical description of LEs for chaotic oscillators. At present time in certain cases for such systems the analytical approximations of Lyapunov exponent were obtained, but these approximations remain valid in a very narrow diapason of the values of the control parameters [24,25] and do not provide a complete picture of the LE behavior.

In this paper we report for the first time on the analytical formula for the so-called zero Lyapunov exponent (ZLE) of chaotic dynamical systems with noise. ZLE stands out among the spectrum of LEs which characterize the complex behavior of chaotic systems. Zero Lyapunov exponent exists necessarily in the LE spectrum of the flow systems characterizing the evolution of the perturbation along the phase trajectory. For two coupled flow oscillators (which possesses two zero LEs in the case without coupling) one of ZLEs diverges from the zero value with the growth of the coupling strength. In addition, ZLE corresponds to the leading Lyapunov exponent of the phase oscillator models [17–20]. It is LE to be the object of the main interest in the present work. ZLE plays a crucial role in some relevant circumstances, e.g., in the synchronization phenomena. In particular, transition of one of ZLEs into the negative value region is related to the phase synchronization phenomenon [26,27], although the transition point does not coincide with the phase synchronization boundary [24,28]. ZLE may also be an indicator of the peculiar regimes of the system behavior, e.g., the incomplete noise-induced synchronization [29].

In parallel with the coupling strength, noise also influences on the LEs. Noise is observed in experimental studies as well as in numerical simulations. Typically, the influence of noise is crucial for the system dynamics (see e.g., Ref. [30–33]).

In addition, the certain phenomena take place in systems with both deterministic (but chaotic) and stochastic dynamics. Indeed, for the driven periodic oscillator behavior the synchronization is known to be connected with the saddle-node bifurcation [34]. The same scenario takes place for chaotic oscillators being in the phase synchronization regime, although its manifestation is hidden due to the aperiodic motion [28,35]. As a consequence, the phenomena observed near the synchronization boundary of periodic oscillators whose dynamics is perturbed by noise have been shown recently to be the same as for the chaotic systems being close to the phase synchronization onset [25,28,32,36]. Similarly, the noise-induced synchronization and generalized synchronization are caused by one and the same mechanism, with the difference between them being only in the driving signal [37]. All findings mentioned above mean that in certain cases the dynamics of chaotic systems may be modeled by the behavior of the periodic systems perturbed by noise. Therefore, the results given in this paper concerning the analytical expression for Lyapunov exponent may be applicable both for the stochastic and deterministic systems as well as for the deterministic systems with noise.

2. Theoretical background

To obtain analytical expression for zero Lyapunov exponent of coupled chaotic oscillators one have to take into account the following points: (i) under certain conditions chaotic oscillators may be modeled by a noised periodic oscillator (see, e.g., [25,37,38]) and (ii) for periodically driven nonlinear oscillator the boundary of synchronization is described by the saddle-node bifurcation [34], with the very same mechanism (but, masked by the irregular dynamics) taking place both for the periodic oscillator perturbed by the external noise and for the chaotic system [25,28]. In other words, to get analytical expression for ZLE, one can consider a model system describing the behavior of driven periodic oscillator with noise in the vicinity of the synchronization onset. From this point of view, the circle map [39–41]

$$\varphi_{n+1} = \varphi_n + \Omega + \varepsilon f(\varphi_n) + \xi_n, \quad \mod 2\pi \tag{1}$$

being a classical model to study nonlinear phenomena [42–44] including synchronization [25] and phase locking [45,46] is the very suitable dynamical system to estimate the value of ZLE of driven periodical oscillator with noise as well as the chaotic oscillators. The circle map (1) is known to describe very precisely the behavior of driven periodical isochronous oscillator near the synchronization onset, since it is, in fact, the discretization of Adler's equation [47] or truncated equation [34,48] deduced in the framework of the complex amplitude method. Tangential bifurcation taking place in the circle map (1) without noise corresponds to the saddle-node bifurcation in the truncated equation and to the synchronization onset of the driven periodical oscillator [34], respectively. The added noise term in (1) enhances the application of the considered map to the noised periodical oscillators and chaotic systems. In Eq. (1) $\varphi \in [0, 2\pi)$ is an angle, parameter $\varepsilon \geq 0$ is a measure of the strength of nonlinearity, ξ_n is a delta-correlated Gaussian noise $[\langle \xi_n \rangle = 0$, $\langle \xi_n \xi_m \rangle = D\delta(n-m)]$, $f(\varphi) = \sin\varphi$. In fact, sin-circle map (1) describes the noised dynamical oscillators driven by an external force whose frequency and amplitude are described by the dimensionless parameters Ω and ε, respectively.

The Lyapunov exponent Λ_0 of circle map (it corresponds to the zero Lyapunov exponent of the coupled flow systems) is

$$\Lambda_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|1 + \varepsilon f'(\varphi_i)|, \tag{2}$$

where $\{\varphi_n\}$ is the time sequence of system (1). Having based on the ergodicity of the examined process and taken into account 2π-periodicity, we can obtain

$$\Lambda_0 = \int_0^{2\pi} \rho_i(\varphi) \ln|1 + \varepsilon f'(\varphi)|\, d\varphi, \tag{3}$$

where

$$\rho_i(\varphi) = \rho_i(\varphi + 2\pi) \tag{4}$$

is the stationary probability density for φ-variable.

Having supposed that φ is changed per one iteration insignificantly, we consider $(\varphi_{n+1} - \varphi_n)$ as the time derivative $\dot{\varphi}$. Due to the stochastic term in (1) we must consider the stochastic differential equation (SDE)

$$d\Phi = (\Omega + \varepsilon \sin\Phi)\, dt + dW, \tag{5}$$