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Analytic self-similar solutions of the Oberbeck–Boussinesq equations

I.F. Barna [a,b,*], L. Mátyás [c]

[a] Wigner Research Center of the Hungarian Academy of Sciences, Konkoly-Thege út 29–33, 1121, Budapest, Hungary
[b] ELI-HU Nonprofit Kft., Dugonics Tér 13, Szeged, 6720, Hungary
[c] Faculty of Science, Sapientia University, Libertătii sq. 1, Miercurea Ciuc, 530104, Romania

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ABSTRACT

In this article we will present pure two-dimensional analytic solutions for the coupled non-compressible Newtonian–Navier–Stokes — with Boussinesq approximation — and the heat conduction equation. The system was investigated from E.N. Lorenz half a century ago with Fourier series and pioneered the way to the paradigm of chaos. We present a novel analysis of the same system where the key idea is the two-dimensional generalization of the well-known self-similar Ansatz of Barenblatt which will be interpreted in a geometrical way. The results, the pressure, temperature and velocity fields are all analytic and can be expressed with the help of the error functions. The temperature field shows a strongly damped single periodic oscillation which can mimic the appearance of Rayleigh–Bénard convection cells. Finally, it is discussed how our result may be related to nonlinear or chaotic dynamical regimes.

1. Introduction

The investigation of the dynamics of viscous fluids has a long past. Enormous scientific literature is available from the last two centuries for fluid motion even without any kind of heat exchange. Thanks to new exotic materials like nanotubes, heat conduction in solid bulk phase (without any kind of material transport) is an other quickly growing independent research area as well. The combination of both processes is even more complex which lacks general existence theorems for unique solutions. The most simple way to couple these two phenomena together is the Boussinesq [1] approximation which is used in the field of buoyancy-driven flow (also known as natural convection). It states that density differences are sufficiently small to be neglected, except where they appear in terms multiplied by g, the acceleration due to gravity. The main idea of the Boussinesq approximation is that the difference in inertia is negligible but gravity is sufficiently strong to make the specific weight appreciably different between the two fluids. When the Boussinesq approximation is used than no sound wave can be described in the fluid, because sound waves move via density variation.

Boussinesq flows are quite common in nature (such as oceanic circulations, atmospheric fronts or katabatic winds), industry (fume cupboard ventilation or dense gas dispersion), and the built environment (like central heating, natural ventilation). The approximation is extremely accurate for such flows, and makes the mathematics and physics much simpler and transparent.

The advantage of the approximation arises because when investigation a flow of, say, warm and cold waters with densities ρ_1 and ρ_2 are considered, the difference $\Delta\rho = \rho_1 - \rho_2$ is negligible and one needs only a single density ρ. It can be shown with the help of dimensional analysis, under these circumstances, the only sensible way that acceleration due to gravity g should enter into the equations of motion is in the reduced gravity $g' = g(\rho_1 - \rho_2)$. The corresponding dimensionless numbers for such flows are the Richardson and

* Corresponding author. Tel.: ++36139222223504.
E-mail address: barna.imre@wigner.mta.hu (I.F. Barna).

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Rayleigh numbers. The used mathematics is therefore much simpler because the density ratio (ρ_1/ρ_2 a dimensionless number) is exactly one and does not affect the features of the investigated flow system.

In the following we analyze the dynamics of a two-dimensional viscous fluid with additional heat conduction mechanism. Such systems were first investigated by Boussinesq [1] and Oberbeck [2] in the 19th century. Oberbeck used a finite series expansion. He developed a model to study the heat convection in fluids taking into account the flow of the fluid as a result of temperature difference. He applied the model to the normal atmosphere.

More than half a century later Saltzman [3] tried to solve the same model with the help of Fourier series. At the same time Lorenz [4] analyzed the solution with computers and published the plot of a strange attractor which was a pioneering results and the advent of the studies of chaotic dynamical systems. The literature of chaotic dynamics is enormous but a modern basic introduction can be found in [5].

Later till to the first beginning years of the millennium [4] Lorenz analyzed the final first order chaotic ordinary differential equation (ODE) system with different numerical methods. This ODE system becomes an emblematic object of chaotic systems and attracts much interest till today [6].

On the other side critical studies came to light which go beyond the simplest truncated Fourier series. Curry for example gives a transparent proof that the finite dimensional approximations have bounded solutions [7]. Roy and Musielak [8] in three papers analyzed large number of truncated systems with different kinds and found chaotic and periodic solutions as well. The messages of these studies will be shortly mentioned later.

In our study we apply a completely different investigation approach, namely the two-dimensional generalization of the self-similar Ansatz which is well-known for one dimension from more than half a century [9–11]. This generalized Ansatz was successfully applied to the three dimensional compressible and non-compressible Navier–Stokes equations [12,13] from us in the last years. We investigated one dimensional Euler equations with heat conduction as well [14] which can be understood as the precursor of the recent study.

To our knowledge this kind of investigation method was not yet applied to the Oberbeck–Boussinesq (OB) system. In the next section we outline our theoretical investigation together with the results. The paper ends with a short summary.

2. Theory and results

We consider the original partial differential equation(PDE) system of Saltzman [3] to describe heat conduction in a two dimensional viscous incompressible fluid. In Cartesian coordinates and Eulerian description these equations have the following form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{\partial P}{\partial x}-\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)=0,$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+\frac{\partial P}{\partial z}-eGT_1-\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)=0,$$

$$\frac{\partial T_1}{\partial t}+u\frac{\partial T_1}{\partial x}+w\frac{\partial T_1}{\partial z}-\kappa\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)=0,$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{1}$$

where u, w, denote respectively the x and z velocity coordinates, T_1 is the temperature difference relative to the average ($T_1=T-T_{av}$) and P is the scaled pressure over the density . The free physical parameters are ν, e, G, κ kinematic viscosity, coefficient of volume expansion, acceleration of gravitation and coefficient of thermal diffusivity. (To avoid further misunderstanding we use G for gravitation acceleration and g which is reserved for a self-similar solution.) The first two equations are the Navier–Stokes equations, the third one is the heat conduction equation and the last one is the continuity equation all are for two spatial dimensions. The Boussinesq approximation means the way how the heat conduction is coupled to the second NS equation. Chandrasekhar [15] presented a wide-ranging discussion of the physics and mathematics of Rayleigh–Benard convection along with many historical references.

Every two dimensional flow problem can be reformulated with the help of the stream function Ψ via $u=\Psi_y$ and $v=-\Psi_x$ which automatically fulfills the continuity equation. The subscripts mean partial derivations. After introducing dimensionless quantities the system of (1) is reduced to the next two PDEs

$$(\Psi_{xx}+\Psi_{yy})_t+\Psi_x(\Psi_{xxz}+\Psi_{yyz})-\Psi_z(\Psi_{xxx}+\Psi_{zzx})$$
$$-\sigma(\theta_x-\Psi_{xxxx}-\Psi_{zzzz}-2\Psi_{xxzz})=0,$$
$$\theta_t+\Psi_x\theta_z-\Psi_z\theta_x-R\Psi_x-(\theta_{xx}+\theta_{zz})=0, \tag{2}$$

where Θ is the scaled temperature, $\sigma=\nu/\kappa$ is the Prandtl number and $R=\frac{GeH^3\Delta T_0}{\kappa\nu}$ is the Rayleigh number and H is the height of the fluid. A detailed derivation of (2) can be found in [3].

All the mentioned studies in the introduction, investigated these two PDEs with the help of some truncated Fourier series, different kind of truncations are available which result different ordinary differential equation (ODE) systems. The derivation of the final non-linear ODE system from the PDE system can be found in the original papers [3,4]. Bergé et al. [16] contains a slightly different development of the Lorenz model equations, and in addition, provides more details on how the dynamics evolve as the reduced Rayleigh number changes. The book of Sparrow [17] gives a detailed treatment of the Lorenz model and its behavior as well. Hilborn [18] presents the idea of the derivation in a transparent and easy way. Therefore, we do not mention this derivation in our manuscript.

Some truncations violates energy conservation [6] and some not. Roy and Musiliak [8] in his exhausting three papers present various energy-conserving truncations. Some of them contain horizontal modes, some of them contain vertical modes and some of them both kind of modes in the truncations. All these models show different features some of them are chaotic and some of them — in well-defined parameter regimes — show periodic orbits in the projections of the phase space. This is a true indication of the complex nature of the original flow problem. It is also clear that the Fourier