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Antiperiodic oscillations in a forced Duffing oscillator

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ABSTRACT

Regularity has always been attributed to periodicity. However, there has been a spurt of interest in another unique type of regularity called anitperiodicity. In this paper we have presented results of antiperiodic oscillations obtained from a forced duffing equation with negative linear stiffness wherein the increase in the number of peaks in antiperiodic oscillation with the forcing strength has been observed. Similarity function has been used to identify the antiperiodic oscillation and further the bifurcation diagram has been plotted and stability analysis of the fixed points have been carried out to understand its dynamics. An analog electronic circuit governed by the forced Duffing equation has been designed and developed to investigate the dynamics of the antiperiodic oscillations. The circuit is quite robust and stable to enable the comparison of its analog output with the numerically simulated data. Power spectrum analysis obtained by fast Fourier transform has been corroborated using a nonlinear statistical technique called rescale range analysis method. By this technique we have estimated the Hurst exponents and detected the coherent frequencies present in the system.

1. Introduction

In the context of nonlinear dynamics, there has been a growing interest in recent years in a particular class of oscillations called antiperiodic oscillations, which posses a unique kind of regularity and periodicity [1,2]. Generally regularity is understood as similar to periodicity, but there exists a relatively unfamiliar type of regularity known as antiperiodic oscillation (APO). An APO, is a special type of periodic oscillation, obeying the relation $x(t+T) = -x(t)$, where T is a time period of antiperiodicity. It is therefore clear that, an antiperiodic oscillation with time period T, will be a periodic oscillation with time period $2T$. The trigonometric functions $\sin(t)$ and $\cos(t)$ are trivial examples of APOs as they follow the identity $\sin(t+\pi) = -\sin(t)$ and $\cos(t+\pi) = -\cos(t)$ respectively, where π is an antiperiod and 2π is a period. These trigonometric functions are the solutions of a harmonic oscillator whose dynamics is governed by the linear differential equations: $\dot{x} = y$ and $\dot{y} = -x$. The linear nature of the harmonic oscillator equation restricts the solution only to a single period oscillation suggesting that nonlinearity may be necessary for the higher period APOs. Theoretical studies on APOs have been carried out in higher order differential equations [3] as well as partial differential equations [4]. Anti periodic solutions have also been explored in heat equation [5], neural networks [6–11], second order Duffing-like oscillators [12], pendulum like [13] oscillators and several other systems [14–16].

The first experimental investigation was carried by Freire et al. [1] in a nonlinear electronic circuit [1] where self excited antiperiodic oscillations were reported and the other was in a Duffing like proxy equation [17]. Recently APOs were reported by Singla et al. [2] in coupled Chua's circuit. In our present work, we have reported observations of APOs in a forced Duffing oscillator which is a low dimensional system as compared to earlier reported experimental results on APOs [1,2,17].

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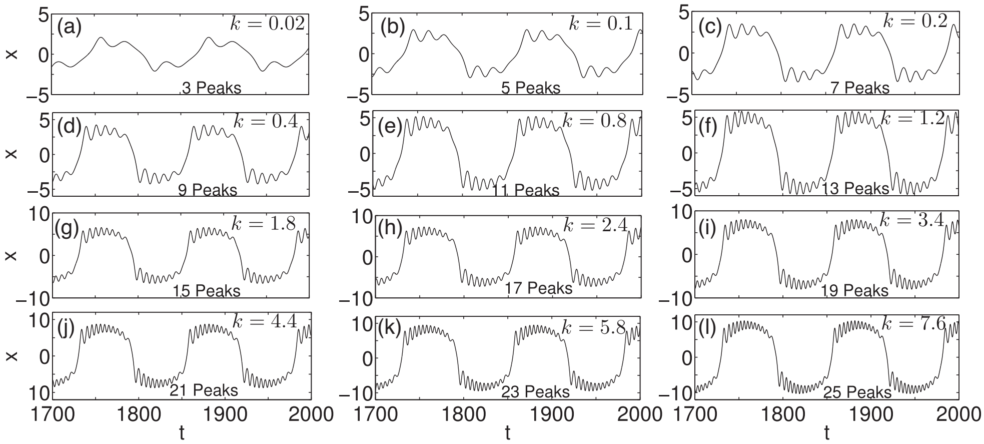

Fig. 1. Plots of numerically simulated antiperiodic time series for the different values of the control parameter k ranging from 0.02 to 7.6.

Forced Duffing equation[18–24] defined in Eq. (1) exhibiting a wide variety of regular and irregular oscillatory patterns is symmetric under parity inversion [25] ($x \to -x, t \to t + \frac{T}{2}$; where T is the period of the driving force). This implies that the output dynamics of a Duffing oscillator could also exhibit antiperiodic property.

In this work, Section 2 contains the numerically simulated time series obtained from Eq. (1), its bifurcation diagram followed by confirmation of APO using the similarity function [26] which is a good tool to identify an APO. In Section 3, the stability of the fixed points and the effect of the forcing term on the dynamics of the equation have been studied. Section 4 contains a discussion on the dynamical mechanism for the generation of APOs. The analytical treatment of the forced Duffing oscillator is carried out in Section 5. Section 6 contains the circuit description of the Duffing oscillator and the experimental results are shown in Section 7. Conclusion and summary of the results are presented in Section 8.

2. The equation of motion and simulated antiperiodic oscillations

A forced Duffing equation with negative linear stiffness, positive damping and periodic excitation, given by

$$\ddot{x} + \delta\dot{x} - \alpha x + \beta x^3 = k\cos(\omega t), \tag{1}$$

where δ (positive) is the damping, k (positive) and ω (positive) are the forcing amplitude and the forcing frequency respectively. Eq. (1) can yield various types of dynamics like periodic, chaotic, spiking and so on depending upon the values of the parameters [20,27].

Eq. (1) can be rewritten as two first order ODEs as

$$\dot{x} = y, \tag{2}$$

$$\dot{y} = -\delta y + \alpha x - \beta x^3 + k\cos(\omega t). \tag{3}$$

The above Eqs. (2) and (3) were solved numerically using the fourth order Runge Kutta method with initial conditions $x_i = 0$ and $y_i = 0$ at $t_i = 0$. The parameters α, β, δ and ω are fixed at 0.01, 0.01, 0.05 and 0.05 respectively, while parameter k is considered as the control parameter.

Fig. 1 shows the numerically simulated time series plots of APOs obtained from Eqs. (2) and (3) for different values of the control parameter k ranging from 0.02 to 7.6. At $k = 0.02$, we observed a clear 3 peak APO and with the increasing k increase in the number of peaks with the following sequence of $3, 5, 7, \ldots$ are observed. APOs were not observed for $k < 0.016$ as seen from the Fig. 2. For $k < 0.004$, the oscillations centered around one of the fixed points (in this case ~1) of the unforced Duffing equation [Fig. 2a]. For $0.004 < k < 0.016$, various types of oscillations like periodic [Fig. 2b and 2d] and unstable APO [Fig. 2e] are observed.

The presence of APOs can be identified using the similarity function [2]. The Similarity function [26] $S(\tau)$ between two time series x_1 and x_2 is defined as

$$S(\tau) = \left(\frac{< [x_1(t) - x_2(t+\tau)]^2 >}{[< x_1^2(t) >< x_2^2(t+\tau) >]^{1/2}} \right)^{1/2}, \tag{4}$$

where τ is the time lag. For $\tau = 0$, S will be some finite positive value (but less than 2) if x_1 and x_2 are different and zero if the two time series are the same. For an APO, the value of $S(\tau)$ will oscillate between 0 and 2 because x_2 is replaced by $-x_1$ in the Eq. (4). Fig. 3 shows the plot of the similarity function between the time series of the variable x and $-x$ for the 3 peak APO (Fig. 1a). In the Fig. 3 it is seen that $S(\tau)$ equals to 2, 0 and 2 for τ equals to 0, 63 and 126 respectively confirming the antiperiodic nature of the oscillation.

The number of peaks as a function of the forcing strength is shown in Fig. 4a. It shows x_{maxima} in the range of the control parameter (forcing strength) $k \in (0, 5)$. Red color dots represent the points where the transition from lower to higher peak APO were found. Figure clearly shows that the number of peaks in APO are increasing with the increase in the forcing strength. An expanded view of the box corresponding to Fig. 4a is shown in Fig. 4b. It shows x_{maxima} in the range of control parameter $k \in (0, 1)$.

The bifurcation diagram, shown in Fig. 5 for the range of the control parameter $k \in (0, 0.02)$, is divided into three different regions: region A ($k < 0.004$), region B ($0.004 < k < 0.016$) and region C ($0.016 < k < 0.02$) respectively. In the region A, the trajectory oscillates around one of the stable attractors of Eq. (2)-(3), whereas in the regions B and C, the oscillations are bounded between the two attractors of Eq. (2)-(3). The trajectories are periodic in the region A and