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Anomalous transport and observable average in the standard map

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A B S T R A C T

The distribution of finite time observable averages and transport in low dimensional Hamiltonian systems is studied. Finite time observable average distributions are computed, from which an exponent α characteristic of how the maximum of the distributions scales with time is extracted. To link this exponent to transport properties, the characteristic exponent $\mu(q)$ of the time evolution of the different moments of order q related to transport are computed. As a testbed for our study the standard map is used. The stochasticity parameter K is chosen so that either phase space is mixed with a chaotic sea and islands of stability or with only a chaotic sea. Our observations lead to a proposition of a law relating the slope in $q = 0$ of the function $\mu(q)$ with the exponent α.

1. Introduction

The question of transport in Hamiltonian systems is a long standing issue as it can be inferred from the vast literature on the matter and references therein [1–7] and the large domain of applications ranging from hot magnetized plasmas to astronomy, chaotic advection, underwater acoustics etc. Beyond the fully chaotic situation in which we usually can apply the central limit theorem, and therefore still have a random walker picture in mind, problems are still not clear when the phase space is mixed.

Indeed in this situation the system is not ergodic, in the sense that there is not only one unique ergodic component, but instead there are regions with chaos, and regions with regular motion. When considering one-and half degrees of freedom system, one usually talks about a picture with a stochastic sea and islands/regions of regular motion. This coexistence can lead to some problems especially since it is possible for Hamiltonian systems to have so called sticky islands. This paper inscribes itself in this series and tries to tackle the problem of transport using distributions of finite time observable averages and their evolution as the average is computed over larger and larger times. A first attempt using this approach was performed in [8], using a perturbed pendulum as a case study, and it found its roots in the study of advection described in [9], where finite time averages were used to detect sticky parts of trajectories. In the present case we simply use the standard or Taylor–Chirikov map [1]. This choice was motivated by the fact that such maps can be directly computed from the flow of the so-called kicked rotor, and, being a map, it allows us to perform fast numerical simulations and gather enough data to have somewhat reliable statistics. The purpose of this paper is not a thorough study of transport in the standard map, but to use this map as a testbed for our analysis of finite time observable averages.

Regarding the problem of transport in Hamiltonian systems, the standard map has become over the years a classical case study. One of its advantages is that it depends on just one control parameter K and many attempts were made to

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find the link between K and a diffusion coefficient [1,5,10,11]. Depending on the values of K, we can get a system which is very close to an integrable one or one that is fully chaotic, with, in between, the picture of a mixed phase space with a chaotic sea and regular islands. In this last setting, we can face the so called stickiness phenomenon: particle's trajectory originating from the chaotic sea can stay (stick) for arbitrary large times in the vicinity of a stable region. This type of phenomenon is able to generate long memory effects, which, in turn, can generate so called anomalous transport also called anomalous diffusion. In contrast with normal diffusion, the dynamics leads to transport properties which can be far from the Gaussian-like processes, and the second moment grows nonlinearly in time.

In the following section, we briefly introduce the standard map and present the phase space and first results with the choice of parameters we considered. Then we discuss and consider transport properties in each system. We present the method and compute characteristic transport exponents. We confirm the multi-fractal nature of transport in both two considered cases where the phase space is mixed, while transport appears as diffusive in the global chaotic case. Finally, we investigate the relation between α, the characteristic exponent of the evolution of the maximum of the distribution of finite-time observable averages, and μ, the characteristic exponent of the second moment of transport associated to the observable. In [8], a simple law was proposed, namely $\alpha = 1 - \mu/2$, our findings lead to good agreements for two out of the three cases. As a consequence, a slightly more general law is then proposed which captures all features; and then we conclude.

2. The standard map

Before moving to more details, we remind the reader that the standard map arises naturally as a Poincaré mapping of the kicked rotor model, whose Hamiltonian writes

$$H = \frac{p^2}{2} - \omega_0^2 \cos(q) \times \sum_{n=-\infty}^{\infty} \delta(t - n\tau)\,, \tag{1}$$

where the parameters ω_0 and τ are without dimensionality and δ is the Dirac function. We shall not derive the standard map here, and we will consider it on the torus. In this case its equations are

$$\begin{cases} p_{n+1} = p_n + K \sin q_n \,[2\pi] \\ q_{n+1} = q_n + p_{n+1} \,[2\pi], \end{cases} \tag{2}$$

where K is the parameter that characterizes the force amplitude [1]. Before going to study transport in this system, we shall briefly present the three different cases considered. Namely, we considered three different values for K. The plots for $K = 1.5$, $K = 2.5$, and $K = 10$ are represented respectively in Figs. 1–3.

Now that we specified the object of our study, let us consider transport.

3. Transport properties

In order to consider transport, we shall first consider an observable. In previous studies [8,9,12,13], it has been found

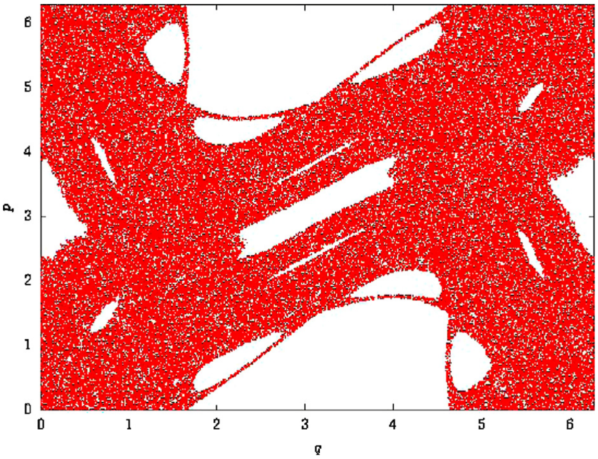

Fig. 1. Phase space visualization for $k = 1.5$. We observe a mixed phase space and Hamiltonian chaos which covers about half of the phase space.

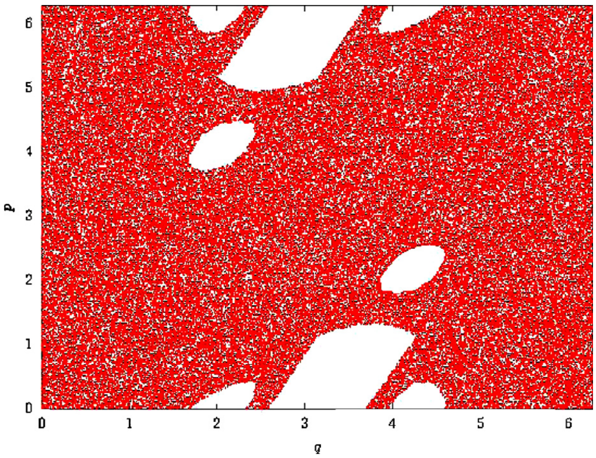

Fig. 2. Phase space visualization for $k = 2.5$. We observe a mixed phase space and Hamiltonian chaos which covers a larger portion than in Fig. 1.

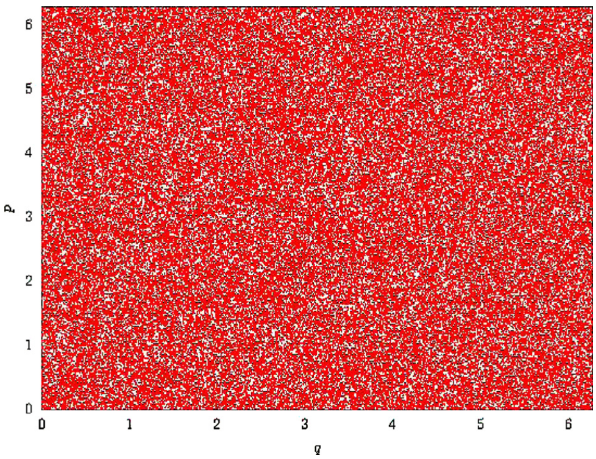

Fig. 3. Phase space visualization for $k = 10$. We observe a fully chaotic phase space and no regular islands subsist.