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Adaptive synchronization of fractional Lorenz systems using a reduced number of control signals and parameters

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ABSTRACT

This paper analyzes the synchronization of two fractional Lorenz systems in two cases: the first one considering fractional Lorenz systems with unknown parameters, and the second one considering known upper bounds on some of the fractional Lorenz systems parameters. The proposed control strategies use a reduced number of control signals and control parameters, employing mild assumptions. The stability of the synchronization errors is analytically demonstrated in all cases, and the convergence to zero of the synchronization errors is analytically proved in the case when the upper bounds on some system parameters are assumed to be known. Simulation studies are presented, which allows verifying the effectiveness of the proposed control strategies.

1. Introduction

The goal of the synchronization of two dynamical systems evolving separately, one called “master” and the other called “slave,” is that those systems will be sharing a common trajectory from a certain time onward. The synchronization of chaotic systems has been widely studied due to its theoretical challenges and its applications in important areas such as secure communications, chemical systems, modeling brain activities [1], ecological systems [2], among others.

The synchronization can be performed under the hypothesis that system parameters are known (nonadaptive synchronization, or simply synchronization) or, if those parameters are unknown (adaptive synchronization) [3]. When the systems to be put in synchrony are described by fractional differential equations, the term fractional adaptive synchronization is used.

We can find in literature many works related to adaptive synchronization, whose results can be applied to the adaptive synchronization of fractional Lorenz systems. Different techniques have been proposed in these works, such as modified projective adaptive synchronization [1,4,5], adaptive full-state linear error feedback [6–8], adaptive sliding mode control [9–12], fuzzy generalized projective synchronization [13], among others [14]. However, these techniques use the maximum possible number of control signals, which in the case of the fractional Lorenz system analyzed in this work is three.

We can find some few works that can be applied to fractional Lorenz system, where only one control signal is used to make adaptive stabilization, using sliding mode control [15–17]. Applying these control techniques it is possible to stabilize a Lorenz system at the origin, using only one control signal. However, the assumption on the system structure for the application of these techniques does not allow their use in synchronization of two fractional Lorenz systems. This is because the definition of the synchronization errors lead to a structure different from the one required for the application of these techniques. Moreover, even for making stabilization of the Lorenz system using these techniques, some of the Lorenz system parameters are needed to construct the control signal, so all the system parameters can not be unknown.

In this paper we study the synchronization of two fractional Lorenz systems with unknown parameters, using a direct approach. The direct approach consists of directly adjusting the controller parameters, without identification of the unknown plant parameters. Since all the parameters of the Lorenz system are considered unknown and only one or two control signals are used to achieve synchronization, this is a work that, as far as the authors know, has not been reported in literature.

Firstly, we analyze the three possible cases where two control signals and one adjustable parameter are used. Next we analyze two cases where only one control signal and one adjustable parameter are employed. In the first four cases studied, no assumptions on the system states boundedness is made. In the fifth case,

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boundedness on the master state trajectories is assumed. The stability of the controlled systems are proved in all cases, using the fractional extension of the Lyapunov direct method, and the main difficulties in proving the convergence to zero of the synchronization errors are exposed.

An alternative solution for the five cases, where an upper bound on some of the system parameters is assumed to be known is presented as well. In these cases, not only the stability of the controlled system is proved, but the convergence to zero of the synchronization errors is proved as well, using the fractional extension of the Lyapunov direct method.

The paper is organized as follows. Section 2 presents some basic concepts of fractional calculus and stability of fractional order systems, which are used along the paper. Section 3 presents the statement of the adaptive synchronization problem, and the proposed solutions in the adaptive case. The alternative solutions assuming a known upper bound on some system parameters are presented as well. The theoretical stability analysis of the controlled system in both cases and the convergence of the synchronization errors when the upper bound on some system parameters are assumed to be known are also presented in Section 3. Section 4 presents the simulation results for the solutions proposed in Section 3, and a comparison with another control strategy available in literature. Finally, Section 5 presents the main conclusions of the work.

2. Some concepts related to fractional calculus and stability of fractional systems

This section presents some basic concepts of fractional calculus and stability of fractional order systems.

2.1. Fractional calculus

Fractional calculus studies integrals and derivatives of orders that can be any real or complex numbers [18]. The Riemann–Liouville fractional integral is one of the main concepts of fractional calculus, and is presented in Definition 1.

Definition 1 (Riemann–Liouville fractional integral [18]). The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ is defined as

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{1}$$

where $t > a$, $\Re(\alpha)$ is the real part of α and $\Gamma(\alpha)$ corresponds to the Gamma Function, given by Eq. (2):

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt. \tag{2}$$

There are some alternative definitions regarding fractional derivatives. Definition 2 corresponds to the fractional derivative according to Caputo, which is the one most frequently used in engineering problems and the one used in this paper.

Definition 2 (Caputo fractional derivative [18]). The Caputo fractional derivative of order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ is defined as

$$_{t_0}^{C}D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{3}$$

where $t > a$, $n = \Re(\alpha) + 1$ for $\alpha \notin \mathbb{N}_0$; $n = \alpha$ for $\alpha \in \mathbb{N}_0$.

2.2. Stability of fractional order systems

The known methods for stability analysis of integer order systems differ from those that have been proposed for fractional order systems. The conditions under which fractional order linear time-invariant systems are stable were studied in [19]. However, in the case of fractional adaptive systems this analysis is not valid, since they are time-varying. The following theorem is used in this paper for the stability analysis of the adaptive fractional synchronization schemes.

Theorem 1 (Lyapunov stability and uniform stability of fractional order systems [20]). *Let $x = 0$ be an equilibrium point for the nonautonomous fractional-order system (4).*

$$_{t_0}^{C}D_t^{\alpha} x(t) = f(x(t), t), \quad \alpha \in (0, 1) \tag{4}$$

Let us assume that there exists a continuous function $V(x(t), t)$ such that

- *$V(x(t), t)$ is positive definite.*
- *$_{t_0}^{C}D_t^{\beta} V(x(t), t)$, with $\beta \in (0, 1]$, is negative semidefinite.*

then the origin of system (4) is Lyapunov stable.

- *Furthermore, if $V(x(t), t)$ is decrescent,*

then the origin of system (4) is Lyapunov uniformly stable.

Besides the stability and uniform stability, asymptotic stability can be proved for fractional order systems using the fractional extension of Lyapunov direct method as well, as it is stated in Theorem 2.

Definition 3. A continuous function γ: $[0, t) \to [0, \infty)$ is said to belong to class-K if it is strictly increasing and $\gamma(0) = 0$ [21].

Theorem 2 (Fractional-order extension of Lyapunov direct method [21]). *Let $x = 0$ be an equilibrium point for the nonautonomous fractional-order system (4). Assume that there exists a Lyapunov function $V(t, x(t))$ and class-K functionsγ_i $(i = 1, 2, 3)$ satisfying*

$$\gamma_1(\|x\|) \le V(t, x(t)) \le \gamma_2(\|x\|) \tag{5}$$

$$_{t_0}^{C}D_t^{\beta} V(t, x(t)) \le -\gamma_3(\|x\|) \tag{6}$$

where $\beta \in (0, 1)$. Then the origin of the system (4) is asymptotically stable.

Remark 1. Given the relationship between positive definite functions and class-K functions, Theorem 2 can be rewritten as in the following.

Let $x = 0$ be an equilibrium point for the nonautonomous fractional-order system (4). Assume that there exists a Lyapunov function $V(t, x(t))$ positive definite and decrescent, satisfying that $_{t_0}^{C}D_t^{\beta} V(t, x(t))$ is negative definite, $\beta \in (0, 1)$, then the origin of system (4) is asymptotically stable.

The following lemma will be useful in proving the stability of fractional synchronization schemes, together with Theorem 1.

Lemma 1 ([20]). *Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable functions. Then, for any time instant $t \ge t_0$, the following relationship holds*

$$\frac{1}{2} {}_{t_0}^{C}D^{\alpha} \left(x^T(t) P x(t) \right) \le x^T(t) P \, {}_{t_0}^{C}D^{\alpha} x(t), \quad \forall \alpha \in (0, 1] \tag{7}$$

where $P \in \mathbb{R}^{n \times n}$ is a constant, square, symmetric and positive definite matrix.

The case when $P = I$ was treated in [22], and the specific scalar case can also be found in [23].