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# Bound the number of limit cycles bifurcating from center of polynomial Hamiltonian system via interval analysis

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ABSTRACT

The algebraic criterion for Abelian integral was posed in (Grau et al. Trans Amer Math Soc 2011) and (Mañosas et al. J Differ Equat 2011) to bound the number of limit cycles bifurcating from the center of polynomial Hamiltonian system. Thisapproach reduces the estimation to the number of the limit cycle bifurcating from the center to solve the associated semi-algebraic systems (the system consists of polynomial equations, inequations and polynomial inequalities). In this paper, a systematic procedure with interval analysis has been explored to solve the SASs. In this application, we proved a hyperelliptic Hamiltonian system of degree five with a pair of conjugate complex critical points that could give rise to at most six limit cycles at finite plane under perturbations $\varepsilon(a+bx+cx^3+x^4)y\frac{\partial}{\partial x}$. Moreover we comment the results of some related works that are not reliable by using numerical approximation.



## 1. Introduction and main results

Hilbert's 16th problem [1] asks for the maximum number (Hilbert number) and distribution of limit cycles that a polynomial planar vector field could have for a given degree $n$. It is unsolved even for the simplest case $n = 2$. Arnold [2] proposed a restricted version which initializes to study small perturbations to Hamiltonian system or planar integrable system as follows:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} + \varepsilon\, p(x, y, \delta) \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} + \varepsilon\, q(x, y, \delta) \end{aligned} \tag{1.1}$$

where $0 \le |\varepsilon| \ll 1$, Hamiltonian $H(x, y)$ and perturbation terms $p(x, y, \delta), q(x, y, \delta)$ are real polynomials in $x$ and $y$, the degree of $H$ is $m$, the degree of perturbation $p, q$ are at most $n$, parameter $\delta \in \mathbb{R}^N$ compact.

Suppose the unperturbed system $(1.1)_{\varepsilon=0}$ has a continuous family of ovals $\gamma_h$ defined by equation $H(x, y) = h, h \in J$, where $J$ is an open interval. Therefore, the following Abelian integral (the first-order Melnikov function)

$$I(h, \delta) = \oint_{\gamma_h} q\,dx - p\,dy, \quad h \in J \tag{1.2}$$

plays a important role in limit cycle bifurcation by perturbing a center. For fixed integers $m$ and $n$, the total number of the limit cycle bifurcating from the period annulus of $(1.1)_{\varepsilon=0}$ is bounded by the maximum $Z(m, n)$ of the number of isolated zeroes of the Abelian integral (1.2), taking into account their multiplicities. If $m = n + 1$, the above problem is usually called the weakened (or tangential, infinitesimal) Hilbert's 16th problem, and the number $\widetilde{Z}(n) = Z(n+1, n)$ can be chosen as the lower bound of Hilbert number $H(n)$.

In the literature, there are many techniques and arguments to tackle the problem of bounding the number of zeroes of Abelian integrals, lots of them are very long and non-trivial, see the part II of [3]. The authors [4,5] proposed a purely algebraic criterion which addresses to verifying the problem whether the collection of Abelian integrals is an ECT-system [4,6–8] or Chebyshev system with accuracy $k$ [5], which implies that the number of real zeroes of any nontrivial linear combination

$$\alpha_0 I_0(h) + \alpha_1 I_1(h) + \cdots + \alpha_{n-1} I_{n-1}(h)$$

is at most $n + k - 1$ counted with multiplicities. The criterion generalizes the work of Li and Zhang [9], which enables to reformulate the problem in a purely algebraic way. By applying the criterion, one could transfer the estimation of the number of real zeroes of Abelian integral to that of the number of real roots of a tuple of associated semi-algebraic systems (SAS for short). By semi-algebraic system, we mean systems consisting of polynomial equations, inequalities and inequations. The reader is referred to see [10,11] for more details. In some sense, this criterion reduces the difficulties of qualitative analysis in limit cycle bifurcating from a center to the computational complexity in estimating the number of real roots of the associated SASs. Hence one keypoint in applying the

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algebraic criterion to Abelian integral is to solve the associated SASs in a practical and efficient algorithm.

Many problems in the computational or applied sciences and engineering can be reduced to the solving of SAS, where the key to the question is to solve polynomial equations. There are two basic approaches to tackle the problem numerically or algebraically. The main methods [12–14] to solve a SAS by symbolic computation are Ritt-Wu method, Gröbner basis method and elimination by resultant, by numerical calculation, there are Newton's iteration and homotopy continuation method and so on. The symbolic computation is strict but sometimes is of low efficiency, whereas the numerical calculation is relatively rapid but the results are sometimes unreliable, a famous cautionary example due to Wilkinson [15], which shows how much the roots of a univariate square-free polynomial can be changed under a small disturbance of its coefficients. Thus, the interval algorithm is applied in the works [7,8,16], and the hybrid algorithm [17] take both the advantages of the symbolic and numerical methods.

Assume that the polynomial equation $W(x,z)=0$ has real root $(\bar{x},\bar{z})$ at the rectangle domain

$$D := \{(x,z) \in R^2 \,|\, z_l < z < 0,\ 0 < x < x_r\},$$

where the variables $x$, $z$ are determined by the equation $q(x,z)=0$, and $z=\sigma(x)$. Here mapping $\sigma$ is an involution, recall an involution is a diffeomorphism with a unique fixed point satisfying $\sigma \neq$ Id and $\sigma^2$=Id.

Our approaches for solving the above SAS is as follows.

1. **(Variable elimination by resultant).** The elimination theory by resultant implies that the components $\bar{x}$, $\bar{z}$ satisfy the resultant equation
$$R(x) = res(q(x,z),\, W(x,z),\, z) = 0,$$
$$\widetilde{R}(z) = res(q(x,z),\, W(x,z),\, x) = 0,$$
respectively.
2. **(Real root isolation).** Isolating the real root of the resultant $R(x)$ (or $\widetilde{R}(z)$) denotes the real root isolation intervals (see Definition 6.2) contained in $(0, x_r)$(or $(x_l, 0)$) by $I_i$ (or $J_k$, respectively), through which to minimize the possible existing domain of the common roots $(\bar{x},\bar{z})$ of $q(x, z)$ and $W_i(x, z)$ in $D$ to the matched boxes $I_i \times J_k$. The theory of real root isolation to univariate polynomial implies in each of them there exists at most one common root.
3. **(The estimation to the number of common solutions).** In each matched box, one could verify whether the box has a common root or not mainly taking advantage of the properties deduced from the involution and the intermediate value theorem of continuous function on compact connected set.

As an application, we consider the perturbations from hyperelliptic Hamiltonian of degree five with a pair of conjugate complex critical points, which is of Liénard equation of type $(4,3)$ as follows:

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x\left(x^2 - x + \dfrac{1}{2}\right)(x+1) + \varepsilon(a + bx + cx^2 + dx^3)y \end{cases} \tag{1.3}$$

with Hamiltonian

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{5}x^5 - \frac{1}{6}x^3 + \frac{1}{4}x^2, \tag{1.4}$$

where $0 < |\varepsilon| \ll 1$, $a, b,\ c$ and $d$ are real bounded parameters.

The closed connected component of a level curve $\{(x,y) \in R^2 |\, \frac{1}{2}y^2 + \frac{1}{5}x^5 - \frac{1}{6}x^3 + \frac{1}{4}x^2 = h, h \in (0, \frac{13}{60})\}$ forms the period orbit $\gamma_h$, the continuous band of which is called period annulus $\mathcal{P}$, the inner boundary is a non-degenerate center $O(0,\ 0)$ and the outer boundary is a homoclinic loop connecting to a hyperbolic saddle $S(-1, 0)$. See Fig. 1.

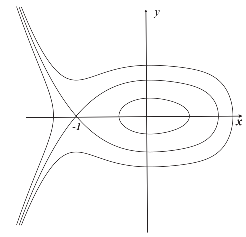


**Fig. 1.** The level curves of system $(1.3)_{\varepsilon=0}$.

With the above approaches to solve the SASs which derive from the Chebyshev criterion to the Abelian integral, we get

**Theorem 1.1.** *For any sufficiently small* $\varepsilon$, *the perturbed Hamiltonian system* (1.3) *could give rise to at most six limit cycle bifurcating from the compact period annulus of system* $(1.3)_{\varepsilon=0}$.

We remark here among the eleven families of hyperelliptic Hamiltonian centers of degree five topologically, the reason why we consider the perturbation $\varepsilon(a+bx+cx^3+x^4)y\frac{\partial}{\partial x}$ from the above center, which bases on a hypothesis that the number of limit cycles by perturbing a center has positive correlation to that of the critical periods assigning to the period annulus. Conjecture 4.1 implies that concerning the weakened Hilbert's 16th problem, in addition to the case the integrable system has more period annuli, it is prior to consider the one that has more critical periods. See Section 4 for more details.

Note that the period annulus of unperturbed system $(1.3)_{\varepsilon=0}$ surrounds a non-degenerate center whose outer boundary is a hyperbolic saddle loop. Roussarie's theorem [18] implies that the upper bound of number of isolated zeroes of $I(h)$ covers the number of limit cycles bifurcating from the non-degenerate center, from the period annuls and from the homoclinic loop connecting to a hyperbolic saddle, therefore we have the following corollary.

**Corollary 1.1.** *For any sufficiently small* $\varepsilon$, *the perturbed Hamiltonian system* (1.3) *could have at most six limit cycles in the finite phase plane enclosing the origin.*

The rest of the paper is organized as follows. Section 2 is devoted to introducing some preliminary definitions and testing methods about Chebyshev properties for Abelian integral, symmetric polynomial. The proof of Theorem 1.1 is given in Section 3, by using the algebraic criterion to Abelian integral proposed in [4,5], a hyperelliptic Hamiltonian of degree five with a pair of conjugate complex critical points could have at most six limit cycles at finite phase plane, the keypoints of the proof is to solve the associated SASs, a complete and practical approach combines with symbolic computation by real-root isolation and numerical computation. Some related works are commented and remarked in Section 4, where we pose a conjecture that the more critical periods of the unperturbed system have, the more limit cycle by perturbing a center might bifurcate. The results of some works on estimating the number of isolated zeroes of Abelian integral is not reliable, due to by using numerical approximation. In Section 5, we comment some drawbacks by applying the algebraic criterion for Abelian integral. Some introductions about the solving of SAS are shown in Appendix.

## 2. Preliminary definitions and methods

In order to prove Theorem 1.1, some definitions and preliminary theorems and lemmas are needed, the reader is referred to [4,5,16] for more details.