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A very efficient approach for pricing barrier options on an underlying described by the mixed fractional Brownian motion

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ABSTRACT

We deal with the problem of pricing barrier options on an underlying described by the mixed fractional Brownian model. To this aim, we consider the initial-boundary value partial differential problem that yields the option price and we derive an integral representation of it in which the integrand functions must be obtained solving Volterra equations of the first kind. In addition, we develop an ad-hoc numerical procedure to solve the integral equations obtained. Numerical simulations reveal that the proposed method is extremely accurate and fast, and performs significantly better than the finite difference method.

1. Introduction

Empirical studies suggest to use the fractional Brownian motion with Hurst exponent $H \in (1/2, 1)$ to model the logarithmic returns of financial assets, the distribution of which is often characterized by self-similarity, heavy tails, long-range dependence and volatility clustering, see, e.g., [1–3]. Nevertheless, even if it is capable of reproducing stylized facts of the stock market returns, the fractional Brownian motion turns out to be problematic when we have to price derivatives. In fact, the classical Itô calculus does not apply to the fractional Brownian motion, as it is neither a Markov process nor a semi-martingale. Thus, unlike what happens in the Black-Scholes market, it is not possible to construct a self-financing strategy yielding the risk-neutral price of financial options. In order to overcome this issue, [4,5] suggest to use the Wick calculus, which allows one to construct Wick-self-financing strategies such that the fractional Black-Scholes market becomes arbitrage free, see, e.g., [4,5]. Notwithstanding, the definition of Wick-self-financing strategy is purely mathematical and does not have any economic interpretation, see, e.g., [6].

Another possible remedy to the shortcomings of the fractional Brownian motion is to model stock returns by the mixed fractional Brownian motion (MFBM, hereafter). The MFBM is a generalization of the fractional Brownian motion obtained as a linear combination of the fractional Brownian motion itself, see, e.g., [7–14] , and of the standard Brownian motion, see, e.g., [15] (for other possible generalizations of the fractional Brownian motion the interested reader is referred to [16–20]).

When the Hurst exponent, H, of the fractional Brownian motion is greater than $\frac{1}{2}$, the MFBM turns out to be a long memory process of Gaussian type. Therefore, the MFBM is particularly suitable for describing the logarithmic returns of the financial assets. Moreover, as proved in [21], when H is taken in the range $\left(\frac{3}{4}, 1\right)$, the MFBM is also arbitrage-free, see, e.g., [22]. For this reason, the MFBM has been employed for pricing standard derivative contracts such as stock options [23–26], currency options [27], equity warrants [28] and credit default swaps [29].

In this paper we consider the problem of pricing barrier options when the underlying asset follows a MFBM. Barrier options are massively traded in the financial markets as they are usually cheaper than standard options, see, e.g., [30,31].

The price of a barrier option can be obtained solving an initial-boundary value partial differential problem in two independent variables. However, such a problem does not have an exact closed-form solution and thus some numerical approximation is required. To this aim, various kinds of discretization techniques could be employed, mainly finite difference methods (see, e.g., [32–35]), finite element methods (see, e.g., [36,37] and reference therein), and variational approaches (see, e.g., [38–41] and references therein). Nevertheless, despite the variety of algorithms available, the one

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proposed in the present paper is the first numerical method developed ad-hoc for pricing barrier options on an underlying described by the MFBM, at least to the best of our knowledge.

In particular, we exploit the mathematical structure of the partial differential problem to be solved and we present an approximation scheme that combines both analytical and numerical techniques. Precisely, first of all, using a variational approach, the partial differential problem is reduced to either a single or a system of two Volterra integral equations of the first kind (depending on whether one or two barriers are considered). Then, the integral equations obtained, which involve only one independent variable, are solved by means of a discretization method based on product integration, see, e.g., [42]. The main advantage of such an approach is that the barrier option price can be easily computed using a direct and fast forward recursion.

The resulting procedure is tested on two different kinds of barrier options: a double-barrier put option and a single-barrier put option with lower barrier. These test cases reveal that the novel approach is extremely efficient from the computational standpoint. In fact, relative errors of order 10^{-6} or even smaller can be obtained in just a few hundredths of a second. Moreover, as shown by numerical experiments reported in Section 6, the proposed integral approximation performs significantly better than the finite difference method.

The reminder of the paper is organized as follows. Section 2 introduces the MFBM and the partial differential problem that allows one to compute barrier option prices. Section 3 outlines the integral formulation of the partial differential problem considered in Section 2. Section 4 performs the numerical approximation of the Volterra integral equations. Section 5 describes how to obtain barrier option prices. Section 6 presents and discusses the results of the numerical simulations. Section 7 concludes.

2. The model

Let us consider an asset whose price satisfies the stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)\big(dB(t) + dB^H(t)\big) \quad (1)$$

with initial condition

$$S(0) = S_0 \quad (2)$$

where μ is a constant drift parameter, σ is a constant volatility parameter, B is a standard Brownian motion and B^H is a fractional Brownian motion with Hurst exponent H. It can be shown that if $H \in (3/4, 1)$ the process (1) and (2) yields long time persistence and absence of arbitrage and thus it is particularly suitable for modeling asset prices.

Let $P(t, S)$ denote the price of a double-barrier put option with maturity T, strike price E, lower barrier S_L and upper barrier S_U on an underlying $S(t)$ described by (1) and (2). Note that we are focusing our attention on a financial option of put type, but the case of a call option is substantially analogous.

Let us set

$$P(t, S) = e^{-r(T-t)}V(t, x), \qquad x = \ln(S) \quad (3)$$

where r is the (constant) spot interest rate, and let us define $x_0 = \ln(S_0)$. According to well-known results in mathematical finance, $V(t, x)$ must satisfy the following partial differential equation for $x \in (x_L, x_U)$, see, e.g., [27]:

$$\frac{\partial V(t,x)}{\partial t} + \left(H\sigma^2 t^{2H-1} + \frac{1}{2}\sigma^2\right)\frac{\partial^2 V(t,x)}{\partial x^2} + \left(r - H\sigma^2 t^{2H-1} - \frac{1}{2}\sigma^2\right)\frac{\partial V(t,x)}{\partial x} = 0 \quad (4)$$

with boundary conditions:

$$V(t, x_L) = 0, \quad V(t, x_U) = 0 \quad (5)$$

and final condition:

$$V(T, x) = \Pi(x) \quad (6)$$

where

$$\Pi(x) = \max\left(e^{x_E} - e^x, 0\right) \quad (7)$$

with $x_L = \ln(S_L)$, $x_U = \ln(S_U)$ and $x_E = \ln(E)$.

We are interested in the solution of problem (4)–(6) at time $t = 0$, which, using the Feynman-Kac theorem (see, e.g., [43]), can be computed as follows:

$$V(0, x_0) = \int_{x_L}^{x_E} \Pi(x) f(0, x_0, T, x)dx \quad (8)$$

where $f(0, x_0, t, x)$ is the probability of having $S(t) = e^x$ and $S_L < S(\tau) < S_U$ $\forall \tau \in [0, t]$ given $S(0) = e^{x_0}$. In the following, in order to keep the notation simple, we write $f(t, x)$ instead of $f(0, x_0, t, x)$ and the dependence of f on x_0 will be understood.

According to well-known results (see, e.g., [43]), the probability density function f can be obtained as the solution of the Kolmogorov forward partial differential equation:

$$\frac{\partial f(t,x)}{\partial t} - \left(H\sigma^2 t^{2H-1} + \frac{1}{2}\sigma^2\right)\frac{\partial^2 f(t,x)}{\partial x^2} + \left(r - H\sigma^2 t^{2H-1} - \frac{1}{2}\sigma^2\right)\frac{\partial f(t,x)}{\partial x} = 0 \quad (9)$$

with boundary conditions:

$$f(t, x_L) = 0, \quad f(t, x_U) = 0 \quad (10)$$

and initial condition:

$$f(0, x) = \delta(x_0) \quad (11)$$

where $\delta(\cdot)$ denotes the Dirac generalized function centered at $\cdot$. The integral method which we propose in this paper in order to compute the barrier option price $V(0, x_0)$ is based on problem (9)–(11).

It is worth pointing out that the price of a single-barrier put option can be obtained solving a partial differential problem that differs from (9)–(11) only for the spatial domain. For example, if we want to price a single-barrier put option we need to solve (9) for $x \in (x_L, +\infty)$, with boundary conditions:

$$f(t, x_L) = 0, \qquad \lim_{x\to+\infty} f(t, x) = 0 \quad (12)$$

and initial condition:

$$f(0, x) = \delta(x_0) \quad (13)$$

Problem (9)–(11) does not have a closed-form solution and thus some kind of numerical approximation is required.

3. The integral formulation

We derive a convenient integral formulation of problem (9)–(11). Let us start by considering two values t_f and x_f, at the moment left unspecified, such that $0 \le t_f \le T$ and $x_L \le x_f \le x_U$. Moreover, let us define

$$\Omega_{t_f} = \left\{(t, x) \in \mathbb{R}^2 \ \middle|\ 0 \le t \le t_f, \ x_L \le x \le x_U\right\} \quad (14)$$

Then, let us consider the function

$$q\big(t, x, t_f, x_f\big) = \frac{1}{\sqrt{2\pi\sigma^2\big(t_f - t + t_f^{2H} - t^{2H}\big)}} e^{-\frac{\left(x_f - x - r\left(t_f - t\right) + \frac{\sigma^2}{2}\left(t_f - t + t_f^{2H} - t^{2H}\right)\right)^2}{2\sigma^2\left(t_f - t + t_f^{2H} - t^{2H}\right)}} \quad (15)$$