Chaos, Solitons and Fractals 87 (2016) 302–306

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Exit location distribution in the stochastic exit problem by the generalized cell mapping method

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ARTICLE INFO

Article history:
Received 4 January 2016
Revised 24 April 2016
Accepted 24 April 2016
Available online 3 May 2016

Keywords:
Exit location distribution
Generalized cell mapping method
Global property
Transient probability matrix

ABSTRACT

The exit location distribution (ELD) in the stochastic exit problem is studied by the generalized cell mapping (GCM) method. According to the global properties of the underlying noise-free system, a proper bounded region is chosen in state space and divided into small cells. The one-step transient probability matrix that governs the global transient short-time solutions of the stochastic system is computed with the consideration of the absorbing boundary condition in exit problem. Based on it, the probability distribution of exit location on domain boundary can be obtained by sufficient evolution of system response starting from the attractor. Two typical examples are given to illustrate the application of the proposed GCM method. It shows that the results obtained by the GCM method agree well with either the results from direct numerical integration or the theoretical predictions.

1. Introduction

In the study of a nonlinear dynamical system perturbed by small external noise, one of the significant tasks is to investigate the stochastic exit problem of the system response from a metastable state. Usually, the basin of attraction of the underlying deterministic system is considered as the safe domain of this metastable state. If the response of noisy system exits from the safe domain ultimately, two questions arise, i.e., when and where? The former is related to the statistics of the first-passage time in stochastic dynamics [1–5]. In recent years, the exit time problem and noise-enhanced stability in nonlinear systems have been studied extensively by Spagnolo and coworkers [6–10].

In order to answer the second question about when, we need to compute an exit location distribution (ELD) [11,12], namely, the probability distribution of exit point on the boundary of domain. It is worth noting that the exit point is the point where the random trajectory crosses the boundary for the first time. During the past decades, the ELD in stochastic exit problem has attracted considerable attentions in many fields, such as statistics physics [13], ecology [14], telecommunications [15], and neuroscience [16]. Two kinds of analytical methods have been used to study the ELD. They are large deviation theory and asymptotic expansion approach [17]. The direct numerical integration offers a common approach. However, it is inherently inefficient when the noise intensity is small.

The generalized cell mapping (GCM) method was firstly developed by Hsu [18]. This new version of cell mapping allows each cell have multiple image cells. The relationship between a cell and its image cells is described by the one-step transition probabilities, which lead to a Markov chain. The GCM method is widely used as an effective tool for global analysis and global bifurcation [19–21] of nonlinear systems. In addition, many attentions have also been paid to the application of GCM method in stochastic and fuzzy systems, such as the studies of random vibration analysis [22], stochastic optional control [23], stochastic bifurcation [24], stochastic response analysis [25,26], fuzzy bifurcation [27], and fuzzy response analysis [28]. The advantage of the GCM method in investigating these problems is that it gives the discrete (both in time axis and state space) global solution of the stochastic system by a short-time transition probability matrix. From this matrix, quantities of useful information can be extracted. With the GCM method, Sun and Hsu [4] successfully studied the first-passage time probability of stochastic dynamical systems, explaining in a probabilistic sense when the responses cross the boundary of a given domain for the first time. Han et al. [5] calculated the probability density of first-passage time and the mean first-passage time from the metastable state in a bistable system subjected to Poisson white noise.

This paper aims to explore another new application of the GCM method in the study of ELD in the stochastic exit problem. The rest of this paper is organized as follows. In Section 2, the GCM method in stochastic dynamical system is reviewed and a variant is specially designed for computing the ELD. Then, in Section 3, two examples are given in detail to illustrate the application of the

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http://dx.doi.org/10.1016/j.chaos.2016.04.017
0960-0779/

proposed GCM method. Finally, some relevant comments are presented in Section 4.

2. GCM method for ELD

2.1. The GCM method

Consider an N-dimensional nonlinear dynamical system subjected to Gaussian white noise excitations. The corresponding stochastic differential equation in the Itô sense is given by

$$\dot{\mathbf{x}}(t) = \mathbf{b}(\mathbf{x}) + \varepsilon^{1/2}\sigma(\mathbf{x})\mathbf{W}(t), \tag{1}$$

where $\mathbf{x}(t)$ is the system response process and it is known to be Markovian. $\mathbf{b}(\mathbf{x})$ is an N-dimensional nonlinear vector function, $\boldsymbol{\sigma}(\mathbf{x})$ is an $N\times L$ diffusion matrix, ε represents a small real parameter, and $\mathbf{W}(t)$ is the standard L-dimensional Gaussian white noise vector process with the following properties

$$E[\mathbf{W}(t)] = \mathbf{0},\ E[\mathbf{W}(t)\mathbf{W}^T(t')] = \mathbf{I}\delta(t-t'),\ \ t' < t, \tag{2}$$

where $\mathbf{0}$ is the L-dimensional zero vector, and $\mathbf{I}$ is the $L\times L$ identity matrix.

Let $q(\mathbf{x}, t|\mathbf{x}_0, t_0)$ denote the transition probability density function (PDF) of the response process $\mathbf{x}$ at time t conditional on $\mathbf{x}_0$ at time t_0 ($t_0 < t$). According to Eqs. (1) and (2), the Fokker–Plank equation [29,30] governing $q(\mathbf{x}, t|\mathbf{x}_0, t_0)$ can be obtained as

$$\frac{\partial q(\mathbf{x},t|\mathbf{x}_0,t_0)}{\partial t} = -\frac{\partial}{\partial x_i}[b_i(\mathbf{x})q(\mathbf{x},t|\mathbf{x}_0,t_0)] + \frac{\varepsilon}{2}\frac{\partial}{\partial x_i\partial x_j}\left[g_{ij}(\mathbf{x})q(\mathbf{x},t|\mathbf{x}_0,t_0)\right], \tag{3}$$

in which the initial condition is $q(\mathbf{x},t|\mathbf{x}_0,t_0) = \delta(\mathbf{x}-\mathbf{x}_0)$, b_i are components of $\mathbf{b}$, and g_{ij} are elements of matrix $\mathbf{g} = \sigma\sigma^T$. Suppose that $p(\mathbf{x}, t)$ denotes the response PDF of the process $\mathbf{x}(t)$ at time t. By the definition of the conditional probability, we have

$$p(\mathbf{x},t) = \int_{R^N} q(\mathbf{x},t|\mathbf{x}_0,t_0)\, p(\mathbf{x}_0,t_0)\mathrm{d}\mathbf{x}_0. \tag{4}$$

When the GCM method is carried out, the continuous state space R^N is firstly transformed into a discrete cell state space Z^N. Consider a bounded region of interest $\mathbf{D}$ and divide it into Nc small rectangular regular cells of the same size. The regular cells are continuously numbered by integers ranging from 1 to Nc. Moreover, the region outside $\mathbf{D}$ is regarded as a sink cell numbered by 0. Once this cell state space is established, the system response is identified within cells, and the transition relationship between points can be replaced by the one between cells.

Now we are only interested in the response process at a sequence of discrete moment $t_n = n\Delta t,\ n = 0,\ 1,\ 2, \ldots$. Choose time $t_0 = m\Delta t$ and $t = (m+1)\Delta t$, and note that the transition PDF is homogeneous, i.e.

$$q(\mathbf{x},t|\mathbf{x}_0,t_0) = q(\mathbf{x},\Delta t|\mathbf{x}_0,0). \tag{5}$$

Then, the transition probability from cell i to j can be expressed as

$$q_{ji} = \int_{C_j} q(\mathbf{x},\Delta t|\mathbf{x}_0,0)\mathrm{d}\mathbf{x}, \tag{6}$$

where $\mathbf{x}_0 \in C_i$, and $C_{i(j)}$ is the domain occupied by cell $i(j)$. If $p_j(n)$ is the probability that the system response locates in the cell j at time t_n

$$p_j(n) = \int_{C_j} p(\mathbf{x},t_n)\mathrm{d}\mathbf{x}. \tag{7}$$

Eq. (4) can be discretized as

$$p_j(n+1) = \sum_{i=1}^{Nc} q_{ji}p_i(n), \quad n = 0, 1, 2, \ldots \tag{8}$$

Assume that $\mathbf{Q} = \{q_{ji}\}$ represents the one-step transition probability matrix and $\mathbf{P}(n) = [p_1(n), p_2(n), \cdots, p_{Nc}(n)]^T$ is the unconditional response probability distribution vector at time t_n. Consequently, Eq. (8) can be rewritten as

$$\mathbf{P}(n+1) = \mathbf{QP}(n). \tag{9}$$

2.2. The ELD

In the GCM method, the one-step transition probability $\mathbf{Q}$ provides the global short-time approximate solutions of the stochastic dynamical system, and Eq. (9) governs the probability evolution of system. This subsection focuses on the computation of ELD in the stochastic exit problem of system (1).

Firstly, the global properties of the underlying deterministic system $\dot{\mathbf{x}}(t) = \mathbf{b}(\mathbf{x})$ should be detected. It is assumed that the system owns an attractor point $\boldsymbol{A}_0$ in the state space, and that $\boldsymbol{B}$, denoting the domain of attraction of $\boldsymbol{A}_0$, is restricted by a smooth boundary $\partial\boldsymbol{B}$. In implementing the GCM method, a bounded region of interest $\mathbf{D}$ is chosen to cover the main global structure in state space, and a cell state space is built in $\mathbf{D}$. Fig. 1(a) presents the global properties in cell state space of a two-dimensional example.

In the case of attraction domain $\boldsymbol{B}$ not confined in state space or too big in size (see Fig. 1(b)), due to the exit point generally distributes on only a part of the smooth boundary $\partial\boldsymbol{B}$, we can select a proper region $\mathbf{D}$ to cover the concerned partial $\partial\boldsymbol{B}$ and part of the domain $\boldsymbol{B}$ (including the attractor point). The partial boundary of $\mathbf{D}$ that locates in domain $\boldsymbol{B}$ is viewed as artificial boundary for considering the exit problem. The artificial boundary should be chosen such that the probability that the system response crosses it is several orders smaller than the probability that the response crosses the concerned partial boundary $\partial\boldsymbol{B}$. In the following, $\boldsymbol{B}$ refers to the part of domain of attraction contained in $\mathbf{D}$. And boundary $\partial\bar{\boldsymbol{B}}$ is the closed curve that consists of the partial $\partial\boldsymbol{B}$ in $\mathbf{D}$ and the artificial boundary.

According to the concerned domain of attraction $\boldsymbol{B}$ and boundary $\partial\bar{\boldsymbol{B}}$, all the cells in cell state space can be classified into three categories. For a given cell c, (i) it is called a boundary cell if there exists a point $x_0 \in c$, and x_0 is located on boundary $\partial\bar{B}$; (ii) it is called a domain cell if $x \in \boldsymbol{B}$ for all $x \in c$; (iii) otherwise, it is called a generalized sink cell.

Secondly, we focus on the construction of a special one-step transition probability matrix $\bar{\mathbf{Q}} = \{\bar{q}_{ji}\}$ with absorbing boundary condition. Let us consider a given cell with number i. If i is a boundary cell or a generalized sink cell, the elements are defined by

$$\bar{q}_{ii} = 1 \text{ and } \ \bar{q}_{ki} = 0 \text{ for all } k \neq i. \tag{10}$$

If i is a domain cell, a counting array NI with initial value 0 is prepared. M interior sampling points are uniformly selected within the cell i. Take each of the sampling points as initial value and generate Sa independent stochastic trajectories for a time duration Δt(mapping time step). In this process, the fourth order Runge–Kutta algorithm is employed as the numerical scheme for integrating the stochastic differential equation. There are a total of $M \times Sa$ random trajectories generated from the considered domain cell i. Suppose that the end point of a random trajectory falls into cell u, two different situations should be taken into account. (i) If u is a domain cell or a boundary cell, the counter $NI(u)$ is increased by one. (ii) If u is a generalized sink cell, this random trajectory must intersect with the boundary $\partial\bar{B}$, and the intersection point is supposed to be in a boundary cell b, then $NI(b)$ is increased by one. After checking all the $M \times Sa$ trajectories from cell i, we can obtain the transition probability from cell i to cell j

$$\bar{q}_{ji} = NI(j)/(M \times Sa). \tag{11}$$