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Predictive control of fractional-order chaotic systems

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ABSTRACT

Stabilizing an unstable equilibrium point in a fractional-order chaotic system is studied using prediction-based feedback method in this paper. Sufficient conditions for stabilization of equilibrium points are explicitly given and rigorously proven. The proposed approach is effective and easy to be implemented, but its application is limited to systems with no external disturbances and uncertainties in the model description. Numerical simulations show the effectiveness of the method.

1. Introduction

Fractional calculus has been introduced since 300 years ago. However, its applications to physics and engineering have just started in the recent decades. It provides an excellent instrument for the description of memory and hereditary properties of various materials and processes, and is widely applied in electromagnetic waves [1], electrode-electrolyte polarization [2], dielectric polarization [3], and viscoelastic systems [4]. Nowadays, it is known that many fractional-order dynamical systems behave chaotically or hyperchaotically, such as the fractional order Chen system [5], the fractional order Lorenz system [6], the fractional-order Duffing system [7], the fractional order Rössler system [8], the fractional order financial system [9], the fractional-order hyperchaotic Chen system [10], the fractional-order hyperchaotic novel system [11] and so on.

The control of chaotic systems has been one of the most interesting topics since Ott, Grebogi and Yorke [12] established an effective scheme to control chaos. Many methods and techniques have been presented to control chaotic systems, such as delayed feedback control method [13], adaptive method [14–16], impulsive method [17], backstepping design technique [18], sliding mode method [19–21], prediction-based feedback method [22], to name but a few. However, most of existing control methods mainly focus on integer-order chaotic systems. The control of fractional-order chaotic systems is still considered as a challenging topic since the theories for analyzing the fractional-order dynamics systems are still very limited.

The prediction-based feedback control method proposed by Ushio and Yamamoto [22] is an efficient method because it don't need any externally generated control signal. In [22], the authors studied the predictive control of integer-order discrete-time chaotic systems where control input was determined by the difference between the predicted states and the current states. Afterwards, Boukabou et al. [23] proposed the predictive control of integer-order continuous-time chaotic systems. Senouci and Boukabou [24] studied the control and synchronization of the integer-order continuous-time chaotic systems based on Takagi–Sugeno fuzzy model and predictive control method. Messadi et al. [25] investigated the predictive control of an integer-order chaotic permanent magnet synchronous generator in a wind turbine system.

In this paper, we investigate predictive control of fractional-order chaotic systems. The main contribution of this paper consists of three aspects. Firstly, we extend prediction-based feedback control method from integer order chaotic systems to fractional-order chaotic systems. The control input is defined as the summation between the predicted uncontrolled state and the difference of actual controlled state and an instable equilibrium point, multiply by a negative controlling parameter. Secondly, the sufficient conditions suppressing fractional-order chaotic systems to unstable equilibrium points are explicitly given and rigorously proven. Thirdly, numerical simulations are presented to verify the results.

The rest of this paper is organized as follows. In Section 2, some preliminaries of fractional derivative are briefly introduced. In Section 3, prediction-based feedback control method of fractional-order chaotic system is presented. Numerical simulation results are provided in Section 4, and conclusions are given in Section 5.

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2. Preliminaries of fractional derivative

There are several definitions of a fractional derivative, such as Grünwald–Letnikov (GL) definition, Riemann–Liouville (RL) definition and Caputo definition. In this paper we mainly use the Caputo fractional derivative. The Caputo fractional derivative of order θ of a continuous function $f: R^+ \to R$ is defined as follows: [26]

$$D^\theta f(t) \equiv \frac{d^\theta f(t)}{dt^\theta} = \begin{cases} \dfrac{1}{\Gamma(m-\theta)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\theta-m+1}} d\tau & m-1 < \theta < m, \\ \dfrac{d^m}{dt^m} f(t) & \theta = m, \end{cases} \tag{1}$$

where $m = [\theta]$ is the first integer which is not less than θ, and $\Gamma(\cdot)$ is the Gamma function and is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \Gamma(z+1) = z\Gamma(z). \tag{2}$$

Consider a general n-dimensional nonlinear fractional-order system

$$\frac{d^\theta x(t)}{dt^\theta} = f(x(t)), \tag{3}$$

where $0 < \theta < 1$ is the fractional order, $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ is state vector of the system, and $f = (f_1, f_2, \ldots, f_n)^T \in R^n$ is differentiable nonlinear vector function. The equilibrium point $x_f \in R^n$ of system (3) is defined by

$$f(x_f) = 0. \tag{4}$$

The Jacobian matrix of the system (3) at the equilibrium points is

$$Df = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{bmatrix}.$$

Lemma 1 ([27]). *If the eigenvalues of the Jacobian matrix Df of the system (3) evaluated at the equilibrium point satisfy*

$$|arg(\lambda)| > \theta\pi/2, \tag{5}$$

namely all the roots of the polynomial equation

$$P(\lambda) = 0, \qquad P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

satisfy Eq. (5), then the system (3) is locally asymptotically stable.

3. Predictive feedback controller design

The purpose of predictive feedback control is to stabilize a fractional-order chaotic system to an equilibrium point by adding a small signal $u(t)$ to the initial system (3). The controlled system will be defined as

$$\frac{d^\theta x(t)}{dt^\theta} = f(x(t)) + u(t). \tag{6}$$

The predictive feedback control law $u(t)$ is determined by the summation between the predicted uncontrolled state and the difference of actual controlled state and an instable equilibrium point, multiply by a negative controlling parameter. It will be in the form

$$u(t) = k\left(\frac{d^\theta x(t)}{dt^\theta} + x(t) - x_f\right) = k(f(x(t)) + x(t) - x_f), \tag{7}$$

where k is a negative controlling parameter such that $k \neq -1$ and x_f is an equilibrium point of system (3). By substituting Eq. (7) into Eq. (6), we obtain the following controlled system:

$$\frac{d^\theta x(t)}{dt^\theta} = \widetilde{f}(x(t)) = (1+k)f(x(t)) + k(x(t) - x_f). \tag{8}$$

It is easy to verify that the controlled system processes the following propositions:

Proposition 1. *The systems (3) and (8) share exactly the same equilibrium point $x_f \in R^n$.*

Proof. If $\widetilde{f}(x_f) = 0$, then $(1+k)f(x_f) = 0$. Then $f(x_f) = 0$ will follow if $k \neq -1$. Conversely, it is obvious that $f(x_f) = 0$ implies $\widetilde{f}(x_f) = 0$. Hence $f(x_f) = 0$ if and only if $\widetilde{f}(x_f) = 0$ and Proposition 1 will follow. □

Proposition 2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the Jacobian matrix Df of the system (3) evaluated at the equilibrium point x_f and $\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_n$ be the eigenvalues of the Jacobian matrix $D\widetilde{f}$ of the system (8) evaluated at the same equilibrium point x_f, there exists the following one-to-one correspondence between their eigenvalues:*

$$\widetilde{\lambda}_j = (1+k)\lambda_j + k, \qquad j = 1, 2, \ldots, n.$$

Proof. It follows from Eq. (8) that

$$D\widetilde{f} = (1+k)Df + kI \tag{9}$$

where I is a unit matrix. Since $\widetilde{\lambda}_j (1 \le j \le n)$ are the eigenvalues of the Jacobian matrix $D\widetilde{f}$ of the system (8), we have $|\widetilde{\lambda}_j I - D\widetilde{f}| = 0$. Then

$$|(\widetilde{\lambda}_j - k)I - (1+k)Df| = (1+k)^n \left|\frac{\widetilde{\lambda}_j - k}{1+k} I - Df\right| = 0,$$

which implies $(\widetilde{\lambda}_j - k)/(1+k) (1 \le j \le n)$ are the eigenvalues of the Jacobian matrix *Df* of the system (3). Hence we have $(\widetilde{\lambda}_j - k)/(1+k) = \lambda_j (1 \le j \le n)$, i.e. $\widetilde{\lambda}_j = (1+k)\lambda_j + k \quad (1 \le j \le n)$. □

From Lemma 1 and Propositions 1 and 2, we obtain the following Theorem.

Theorem 1. *Let α_j and β_j be the real parts and imaginary parts of the eigenvalues $\lambda_j (j = 1, 2, \ldots, n)$ of the Jacobian matrix Df of the system (3) evaluated at the equilibrium point $x_f \in R^n$, respectively*

$$\bar{k} = \min_{1 \le j \le n} \left\{ \frac{-\alpha_j \tan\frac{\theta\pi}{2} + |\beta_j|}{(1+\alpha_j)\tan\frac{\theta\pi}{2} - |\beta_j|} \,\Big|\, |arg(\alpha_j + \beta_j i)| \le \frac{\theta\pi}{2} \right\}. \tag{10}$$

If $k \in (-1, \bar{k})$, then the equilibrium point $x_f \in R^n$ of the controlled system (8) is locally asymptotically stable.

Proof. Since $|arg(\alpha_j + \beta_j i)| \le \frac{\theta\pi}{2}$ and $0 < \theta < 1$, we have $\alpha_j > 0$ and $\arctan\frac{|\beta_j|}{\alpha_j} \le \frac{\theta\pi}{2}$. Hence $\frac{-\alpha_j \tan\frac{\theta\pi}{2} + |\beta_j|}{(1+\alpha_j)\tan\frac{\theta\pi}{2} - |\beta_j|} \le 0$, which implies $\bar{k} \le 0$. According to the Proposition 2, the real part and imaginary part of eigenvalue $\widetilde{\lambda}_j$ of the Jacobian matrix $D\widetilde{f}$ of the controlled system (8) are $(1+k)\alpha_j + k$ and $(1+k)\beta_j$, respectively. It follows from Lemma 1 that the locally asymptotically stable condition for the system (8) is

$$|arg((1+k)\alpha_j + k + (1+k)\beta_j i)| > \frac{\theta\pi}{2}. \tag{11}$$

The proof proceeds in the following cases.

Case 1: Suppose that $\alpha_j \le 0$, then $k \in (-1, \bar{k})$ allows us to infer the stable condition (11).

Case 2: Suppose that $\alpha_j > 0$, then it is obvious that $(1+k)\alpha_j + k \le 0$ implies the stable condition (11) for $k \in (-1, \bar{k})$.