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Continuum-wise expansiveness for non-conservative or conservative systems

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ABSTRACT

In this paper, we show that a non-conservative vector field is robustly continuum-wise expansive if and only if it satisfies both Axiom A and the quasi-transversality condition. Moreover, a conservative vector field (divergence-free vector field, Hamiltonian system) is robustly continuum-wise expansive if and only if it is Anosov.

1. Introduction

Let M be a closed smooth $n(n \geq 2)$-dimensional Riemannian manifold, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1-topology. A main research topic on differentiable dynamical systems is the hyperbolic structure such as Axiom A, Anosov, structural stability, etc. We say that f is *expansive* if there is $e > 0$ such that for any pair of distinct points $x, y \in M$ there is $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > e$. Expansiveness is a useful concept to study the hyperbolic structure.

In fact, Mañé [21] proved that if a diffeomorphism belongs to the C^1 interior of the set of all expansive diffeomrphisms then it is quasi-Anosov. Here a diffeomorphism f is *quasi-Anosov* if for all $v \in TM\setminus\{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded. The expansiveness was introduced by Utz [31]. After, many researchers introduce the definitions of various expansiveness (N-expansive [22], entropy expansive [9], measure expansive [23], continuum-wise expansive [15], etc).

Also, the definitions are closely related to the hyperbolic dynamics ([19,29,30]). Usually, the results of discrete dynamical system (diffeomorphism) can be extended to the case of continuous dynamical system (vector field, or flow). However, the results of vector fields cannot be obtained directly from that of diffeomorphisms in general. A diffeomorphism f: $M \to M$ is called a *star diffeomorphisms* if f has a C^1 neighborhood $\mathcal{U}(f)$ such that every periodic point of every $g \in \mathcal{U}(f)$ is hyperbolic. Aoki [2] and Hayashi [14] proved that if a diffeomorphism f is a star diffeomorphism then f satisfies Axiom A without cycles. A vector field X is called a *star vector field* (or a flow X^t is called a *star flow*) if a vector field X has a C^1 neighborhood $\mathcal{U}(X)$ such that every singularity and every periodic orbit of every $Y \in \mathcal{U}(X)$ is hyperbolic. Then we can find that if a flow X^t is a star flow then it satisfies Axiom A but it does not satisfy the no-cycle condition [20]. The flows version of expansiveness was introduced and studied by [10] (the definition is in the next section). Moriyasu *et al.* [25] studied that the version for flows of the result of Mañé [21]. Which is a motivation of the paper. We study the continuum-wise expansive flow (the definition is in the next section) which is a generalization of the results of [4,25].

2. Non-conservative vector fields

Let M be a closed $n(n \geq 3)$-dimensional C^∞ Riemannian manifold, and let d be the distance on M induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle TM. Denote by $\mathfrak{X}(M)$ the set of all C^1-vector fields on M endowed with the C^1-topology. Let $X \in \mathfrak{X}(M)$. Then X generates a C^1 flow $(X^t)_{t\in\mathbb{R}}$ on M. Let $X^t : \mathbb{R} \times M \to$

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M be a flow. Then X^t satisfies the followings: (i) $X^0(x) = x$, for all $x \in M$, and (ii) $X^s(X^t(x)) = X^{s+t}(x)$ for all $x \in M$ and all $s, t \in \mathbb{R}$. For $x \in M$, the $Orb(x) = \{X^t(x) : t \in \mathbb{R}\}$ is called the orbit of X through x. A point x is *non-wandering* if for every $\tau > 0$ and every neighborhood U of x there is $t > \tau$ such that $X^t(U) \cap U \neq \emptyset$. Denote by $\Omega(X)$ the set of all non-wandering points of X.

Given a vector field X we denote by $Sing(X)$ the set of *singularities* of X, i.e. those points $x \in M$ satisfies $X(x) = \vec{0}$. Let $R := M \backslash Sing(X)$ be the set of *regular* points. We say that X^t is *expansive* if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in M$ there is an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that if $d(X^t(x), X^{h(t)}(y)) < \delta$ for $t \in \mathbb{R}$ then $y \in X^{(-\epsilon,\epsilon)}(x)$. By Oka [27, Lemma 2], if a flow X^t is expansive then $Sing(X) = \emptyset$.

Let Λ be a closed X^t-invariant set. The set Λ is called *hyperbolic* for X^t if there are constants $C > 0$, $\lambda > 0$ and a splitting $T_xM = E^s_x \oplus \langle X(x)\rangle \oplus E^u_x$ such that the tangent flow DX^t: $TM \to TM$ leaves the invariant continuous splitting and

$$\|DX^t|_{E^s_x}\| \le Ce^{-\lambda t} \quad \text{and} \quad \|DX^{-t}|_{E^u_x}\| \le Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$. We say that $X \in \mathfrak{X}(M)$ is *Anosov* if M is hyperbolic for X.

We say that X satisfies *Axiom A* if the non-wandering set $\Omega(X)$ is hyperbolic and $P(X) \cup Sing(X)$ are dense in $\Omega(X)$, where $P(X)$ is the set of periodic points of X.

Denote by $\mathcal{G}^*(M)$ the set of all-star vector fields. In [13] Gan and Wen proved that if a non-singular vector field $X \in \mathcal{G}^*(M)$ then it satisfies both the periodic orbits are dense in $\Omega(X)$ and $\Omega(X)$ is hyperbolic. For any hyperbolic point $x \in M$, we define the sets $W^s(x) = \{y \in M : d(X^t(x), X^t(y)) \to 0 \text{ as } t \to \infty\}$ and $W^u(x) = \{y \in M : d(X^t(x), X^t(y)) \to 0 \text{ as } t \to -\infty\}$, where $W^s(x)$ is called the *stable manifold* of the point x and $W^u(x)$ is called the *unstable manifold* of the point x. Let $X \in \mathfrak{X}(M)$ satisfy Axiom A. We say that X satisfies the *quasi-transversality condition* if $T_xW^s(x) \cap T_xW^u(x) = \{O_x\}$ for all $x \in M$.

Continuum-wise expansiveness was introduced by Kato [15]. Recently, Arbieto, Cordeiro and Pacifico [3] was introduced the corresponding concept for flow. Let $Hom(\mathbb{R}, 0)$ be the set of homeomorphisms on $\mathbb{R}$ fixing the origin and if A is a subset of M, $C^0(A, \mathbb{R})$ denotes the set of real continuous maps defined on A. Define $\mathcal{H}(A) = \{\alpha : A \to Hom(\mathbb{R}, 0) : \text{there is } x_\alpha \in A \text{ with } \alpha(x_\alpha) = id \text{ and } \alpha(.)(t) \in C^0(A, \mathbb{R}) \text{ for all } t \in \mathbb{R}\}$, and if $t \in \mathbb{R}$ and $\alpha \in \mathcal{H}(A)$, $\mathcal{X}^t_\alpha(A) = \{X^{\alpha(x)(t)}(x) : x \in A\}$.

Definition 2.1 ([3, Definition 1.1]). Let $X \in \mathfrak{X}(M)$. We say that a flow X^t is continuum-wise expansive if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $A \subset M$ is a continuum and $h \in \mathcal{H}(A)$ satisfies

$$\text{diam}(\mathcal{X}^t_h(A)) < \delta, \text{ for all } t \in \mathbb{R}$$

then $A \subset X^{(-\epsilon,\epsilon)}$.

Definition 2.2. Let $X \in \mathfrak{X}(M)$. We say that a flow X^t is robustly continuum-wise expansive if there is a C^1 neighborhood $\mathcal{U}(X)$ such that for any $Y \in \mathcal{U}(X)$, Y^t is continuum-wise expansive.

Moriyasu et al. [25] proved that if a vector field X belongs to the C^1 interior of the set of all expansive vector fields then it satisfies both Axiom A and quasi-transversality condition. From this fact, we consider a general result as follows.

Theorem A. *Let $X \in \mathfrak{X}(M)$. A flowX^t is robustly continuum-wise expansive if and only ifX^t satisfies both Axiom A and the quasi-transversality condition.*

3. Conservative systems-divergence free vector fields

Let M be Riemannian closed and connected manifold $n(n \ge 3)$-dimensional and let $d(\cdot, \cdot)$ denote the distance on M inherited by the Riemannian structure. We endow M with a volume-form [26] and let μ denote the Lebesgue measure related to it. Given a C^r ($r \ge 1$) vector field X: $M \to TM$ the solution of the equation $x' = X(x)$ generates a C^r flow, X^t; by the other side given a C^r flow we can define a C^{r-1} vector field by considering $X(x) = \frac{dX^t(x)}{dt}|_{t=0}$. We say that X is *divergence-free* if its divergence is equal to zero, that is, $\nabla \cdot X = 0$ or equivalently if the measure μ is invariant for the associated flow. Let $\mathfrak{X}_\mu(M)$ denote the space of C^r divergence-free vector fields and we consider the usual C^1 Whitney topology on this space. Lee [19] proved that if a C^1 volume preserving diffeomorphism f is robustly continuum-wise expansive then it is Anosov. For divergence-free vector fields, Ferreira [12] proved that if a divergence-free vector field X belongs to the C^1 interior of the set of all expansive divergence free vector fields then it is Anosov. Bessa *et al.* [5] proved that for C^1 generic sense if a divergence free vector field X is expansive then it is Anosov. From the results, we consider the following theorem which is a general result for expansiveness of divergence free vector fields.

Theorem B. *Let $X \in \mathfrak{X}_\mu(M)$. If a flow X^t is robustly continuum-wise expansive then it is Anosov.*

4. Conservative systems-Hamiltonian systems

Let (M, ω) be a symplectic manifold, where M is a $2n(\ge 2)$-dimensional, compact, boundaryless, connected and smooth Riemannian manifold, endowed with a symplectic form ω. A *Hamiltonian* $H : M \to \mathbb{R}$ is a real valued $C^r (r \ge 2)$ function on M. Denote by $C^r(M, \mathbb{R})$ the set of C^r-Hamiltonians on M. In this paper, we will be restricted to the C^2-topology, thus we set $r = 2$. Given a Hamiltonian H, we define the *Hamiltonian vector field* X_H as following: for all $v \in T_pM$

$$\omega(X_H(p), v) = d_pH(v),$$

which generates the Hamiltonian flow X^t_H. A Hamiltonian vector field X_H is C^1 if and only if the Hamiltonian function H is C^2. A scalar $e \in H(M) \subset \mathbb{R}$ is called an *energy* of H. An *energy hypersurface* $\mathcal{E}_{H,e}$ is a connected component of $H^{-1}(\{e\})$ called an *energy level set*. The energy level set $H^{-1}(\{e\})$ is said to be *regular* if any energy hypersurface of $H^{-1}(\{e\})$ is regular which means it does not contain singularities. Clearly a regular energy hypersurface is a X^t_H-invariant, compact and $(2n-1)$-dimensional manifold. We say that a Hamiltonian level (H, e) is *regular* if the energy level set $H^{-1}(\{e\})$ is regular. A *Hamiltonian system* is $(H, e, \mathcal{E}_{H,e})$, where H is a Hamiltonian, e is an energy and $\mathcal{E}_{H,e}$ is a regular connected component of $H^{-1}(\{e\})$. Then $H^{-1}(\{e\})$ corresponds to the union of a finite number of closed connected components, that is, $H^{-1}(\{e\}) = \bigcup_{i=1}^n \mathcal{E}_{H,e,i}$, for $n \in \mathbb{N}$. Note that it is well known that Hamiltonian flows are symplectic and volume preserving by the Liouville Theorem. Thus the $2n$-form $\omega^n = \omega \wedge \ldots \wedge \omega(n\text{-times})$ is a volume form and induces a measure μ on M which is called the Lebesgue measure associated to ω^n. Then the measure μ on M is invariant by the Hamiltonian flow. For symplectic diffeomorphisms, Lee [17] showed that if a symplectic diffeomorphism f is robustly continuum-wise expansive then it is Anosov. Moreover, the author [18] proved that for C^1 generically, if a symplectic diffeomorphism f is continuum-wise expansive then it is Anosov. Bessa *et al.* [8] proved that a Hamiltonian system $(H, \mathcal{E}, e)$ is robustly expansive then it is Anosov. Lee [16] proved that for C^2 generically, if a Hamiltonian system $(H, \mathcal{E}, e)$ is expansive then it is Anosov. From the results, we consider the following.

Theorem C. *Let $H \in C^2(M, \mathbb{R})$. If a Hamiltonian system $(H, \mathcal{E}, e)$ is robustly continuum-wise expansive then it is Anosov.*