Chaos, Solitons and Fractals 83 (2016) 18–26

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Fatness and thinness of some general Cantor sets for doubling measures☆

Jiaojiao Yang*

Department of Mathematics, South China University of Technology, Guangzhou 510641, PR China

ARTICLE INFO

Article history:
Received 1 June 2015
Accepted 21 November 2015
Available online 11 December 2015

MSC:
28A80
28A33

Keywords:
(α_n)-regular sets
Doubling measure
Fat set
Thin set

ABSTRACT

In this paper, we consider some classes of general Cantor sets satisfying (α_k)- regular condition and obtain necessary and sufficient conditions to characterize the fatness and thinness of the sets for doubling measures under suitable conditions. It is applied to classify some kinds of Cantor-type sets.

1. Introduction

A Borel probability measure μ on [0, 1] is doubling if there is a constant $K \geq 1$ such that

$$\mu(I) \leq K\mu(J) \tag{1.1}$$

for all pairs I, J of adjacent intervals of equal length in [0, 1]. In this case we also say that μ is K-doubling on [0, 1]. Denote by $\mathcal{DM}_K[0,1]$, the set of K-doubling measures on[0, 1] and by $\mathcal{DM}[0,1]$, the set of doubling measures on [0, 1]. It is clear that $\mathcal{DM}_{K_1}[0,1] \subset \mathcal{DM}_{K_2}[0,1]$ if $K_1 < K_2$, and that $\mathcal{DM}[0,1] = \bigcup_{K \geqslant 1} \mathcal{DM}_K[0,1]$. For some results on doubling measures we refer to [5,8,11].

Let $E \subset [0,1]$, we say that E is fat if $\mathcal{L}(E) > 0$, and that E is thin if $\mathcal{L}(E) = 0$, where $\mathcal{L}$ denotes the Lebesgue measure. According to the relationship between fat (thin) sets and doubling measure on [0, 1], a fat set E can be classified to three classes as follows:

(i) E is very fat if $\mu(E) > 0$ for any $\mu \in \mathcal{DM}[0,1]$.
(ii) E is fairly fat if there are constants $1 < K_1 < K_2$ such that $\mu(E) > 0$ for any $\mu \in \mathcal{DM}_{K_1}[0,1]$, but $\mu(E) = 0$ for some $\mu \in \mathcal{DM}_{K_2}[0,1]$.
(iii) E is minimally fat if for every $K > 1$ there is $\mu \in \mathcal{DM}_K[0,1]$ such that $\mu(E) = 0$.

Similarly, a thin set E can be classified to three classes: E is very thin if $\mu(E) = 0$ for any $\mu \in \mathcal{DM}[0,1]$; E is fairly thin if there are constants $1 < K_1 < K_2$ such that $\mu(E) = 0$ for any $\mu \in \mathcal{DM}_{K_1}[0,1]$, but $\mu(E) > 0$ for some $\mu \in \mathcal{DM}_{K_2}[0,1]$; E is minimally thin if for every $K > 1$ there is $\mu \in \mathcal{DM}_K[0,1]$ such that $\mu(E) > 0$.

It is clear that a subset of [0, 1] with nonempty interior is fat and that a countable subset is thin. Moreover, a set of Hausdorff dimension zero is very thin (see [4]). According to Tukia [10], a set of full Lebesgue measure may not be very fat and for any $\varepsilon \in (0,1)$, a set of Hausdorff dimension ε may not be very thin. However, Wu [12] showed that an (α_k)-porous

☆ This work is supported by NSFC (grant no. 11371148) and Fundamental Research Funds for the Central Universities, SCUT (2015ZZ127).

* Tel.: +86 13763352954.
E-mail address: y.jj@mail.scut.edu.cn

http://dx.doi.org/10.1016/j.chaos.2015.11.032

set with $\sum_{k=1}^{\infty}{\alpha_k}^p=\infty$ for all $p>0$ is very thin. Staples and Ward [9] proved an (α_k)-thick set with $\sum_{k=1}^{\infty}{\alpha_k}^p<\infty$ for all $p>0$ is very fat.

Later, Buckley et al. [1] studied the fatness of middle interval Cantor sets. In [3], Han et al. discussed the fatness and thinness of uniform Cantor sets $E=E(\{n_k\},\{c_k\})$. They proved that the uniform Cantor set E is very fat if and only if $\sum_{n=1}^{\infty}(n_kc_k)^p<\infty$ for all $0<p<1$ and E is very thin if and only if $\sum_{n=1}^{\infty}(n_kc_k)^p=\infty$ for all $p>1$. Further, Peng and Wen [7] characterized the fairly (minimally) fat (thin) uniform Cantor sets. In this paper, we shall consider the fatness and thinness for a class of general Cantor sets.

Now we define general Cantor sets. Let $\{n_k\}_{k\geq 1}\subset\mathbb{N}$ be a sequence satisfying $n_k\geq 2$, for any $k\geq 1$. Set $\Omega_k=\Big\{\sigma=i_1\ldots i_k: i_j\in\{1,2,\ldots,n_j\}$ for all $1\leq j\leq k\Big\}$ and $\Omega_0=\{\emptyset\}$. Write $\Omega=\bigcup_{k\geq 0}\Omega_k$ and $(i_1\ldots i_k)*i_{k+1}=i_1\ldots i_ki_{k+1}$. Denote by $|A|$ the diameter of $A\subset[0,1]$ and $int(A)$ denotes the interior of the set A. We say that a collection $\mathcal{F}=\{I_\sigma\}_{\sigma\in\Omega}$ of closed subintervals of [0, 1] has general Cantor structure, provided that $I_\emptyset=[0,1]$ and for any $k\geq 1$ and $\sigma\in\Omega_{k-1}$, $I_{\sigma*1},I_{\sigma*2},\cdots,I_{\sigma*n_k}$, located from left to right, are subintervals of I_σ with $int(I_{\sigma*i})\bigcap int(I_{\sigma*j})=\emptyset$ for $i\neq j$ and $sup_{\sigma\in\Omega_k}|I_\sigma|\to 0$ as $k\to\infty$.

The general Cantor set associated with $\mathcal{F}$ is defined by

$$C=C(\mathcal{F})=\bigcap_{k\geq 1}\bigcup_{\sigma\in\Omega_k}I_\sigma, \tag{1.2}$$

where any I_σ of $\mathcal{F}$ is called a basic interval of rank k for $\sigma\in\Omega_k$.

The general Cantor set C is said to be *end-point regular* if for all $k\geq 1$ and $\sigma\in\Omega_{k-1}$, if $I_\sigma=[a,b]$, the left endpoint of $I_{\sigma*1}$ is a, the right endpoint of $I_{\sigma*n_k}$ is b.

Remark 1. The uniform Cantor set in [3] and the Moran sets in [6] are all special classes of end-point regular general Cantor sets.

For all $k\geq 1,\sigma\in\Omega_{k-1}$ and $1\leq i\leq n_k$, write $I_{\sigma*i}=[a_i^\sigma,b_i^\sigma]$. For all $1\leq j\leq n_k-1$, let

$$J_{\sigma,j}=\begin{cases}(b_j^\sigma,a_{j+1}^\sigma), & \text{if } dist(I_{\sigma*j},I_{\sigma*(j+1)})>0,\\ \emptyset, & \text{if } dist(I_{\sigma*j},I_{\sigma*(j+1)})=0,\end{cases}$$

where dist (A,B) denotes the distance between two sets A and B.

Denote $\mathcal{I}_k=\{I_\sigma\}_{\sigma\in\Omega_k}$, $\mathcal{J}_k=\{J_{\sigma,j};\ J_{\sigma,j}\subset I_\sigma\}_{\sigma\in\Omega_{k-1},1\leq j\leq n_k-1}$, $\mathcal{J}=\bigcup_{k\geq 1}\mathcal{J}_k$, where any $J_{\sigma,j}$ of $\mathcal{J}$ is called a basic gap of rank $|\sigma|+1$. It is easy to see that $\mathcal{F}=\bigcup_{k\geq 0}\mathcal{I}_k$, $I_\sigma=(\bigcup_{1\leq i\leq n_k}I_{\sigma*i})\cup(\bigcup_{1\leq j\leq n_k-1}J_{\sigma,j})$ for all $k\geq 1,\sigma\in\Omega_{k-1}$.

Definition 1. Given a sequence $\{\alpha_k\}_{k=1}^{\infty}$ with $0<\alpha_k<1$, the end-point regular general Cantor set C is said to satisfy (α_k)-**regular condition**, if there are constants $0<r_2\leq r_1<\infty$ such that $r_2\alpha_k|I_\sigma|\leq|J_{\sigma,j}|\leq r_1\alpha_k|I_\sigma|$, for any $k\geq 1,\sigma\in\Omega_{k-1},1\leq j\leq n_k-1$.

Definition 2.

(1) The end-point regular general Cantor set C is said to be good, if there exists $c\in(0,1)$ so that $cI_\sigma\bigcap J_{\sigma,j}\neq\emptyset$, for all $k\geq 1,\sigma\in\Omega_{k-1},1\leq j\leq n_k-1$.

(2) The end-point regular general Cantor set C is said to be rarely good, if there exists $c\in(0,1)$ so that $J_{\sigma,j}\subset cI_\sigma$, for all $k\geq 1,\sigma\in\Omega_{k-1},1\leq j\leq n_k-1$, where cI_σ denotes the interval that is concentric with I_σ and with length $c|I_\sigma|$.

For $0<p<\infty$, denote $l^p=\{(\alpha_n);\sum_{n=1}^{\infty}{\alpha_n}^p<\infty\}$. Now we state our main results in this paper.

Theorem 1. *Let C be a good general Cantor set satisfying (α_n)-regular condition and $M=\sup_k\{n_k\}<\infty$. Then*

(1) *C is very fat if and only if $(\alpha_n)\in\bigcap_{0<p<1}l^p$.*
(2) *C is very thin if and only if $(\alpha_n)\notin\bigcup_{p>1}l^p$.*

Theorem 2. *Let C be a rarely good general Cantor set satisfying (α_n)-regular condition and $M=\sup_k\{n_k\}<\infty$.*
If C is fat, then

(1) *C is minimally fat if and only if $(\alpha_n)\notin\bigcup_{0<p<1}l^p$.*
(2) *C is fairly fat if and only if $(\alpha_n)\notin\bigcap_{0<p<1}l^p$ and $(\alpha_n)\in\bigcup_{0<p<1}l^p$.*
If C is thin, then
(3) *C is minimally thin if and only if $(\alpha_n)\in\bigcap_{p>1}l^p$.*
(4) *C is fairly thin if and only if $(\alpha_n)\notin\bigcap_{p>1}l^p$ and $(\alpha_n)\in\bigcup_{p>1}l^p$.*

If the general Cantor set C satisfies the lengths of basic intervals (respectively, gaps) with the same rank are comparable, that is to say, for any $k\geq 1,\sigma\in\Omega_{k-1}$, there exist constants $l_k\geq 1$, $m_k\geq 1$ such that $\frac{1}{l_k}|I_{\sigma*i}|\leq|I_{\sigma*i'}|\leq l_k|I_{\sigma*i}|$, $1\leq i,i'\leq n_k$ and $\frac{1}{m_k}|J_{\sigma,j}|\leq|J_{\sigma,j'}|\leq m_k|J_{\sigma,j}|$, $1\leq j,j'\leq n_k-1$, then the general Cantor set C is said to satisfy **comparable condition**, where $\{l_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ are called control sequences.

Theorem 3. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence with $a_k\in(0,1)$, $n_ka_k<1$ for all $k\geq 1$ and C is the general Cantor set satisfying (a_k)-regular condition and comparable condition with bounded control sequences $\{l_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$.*
If C is fat, then

(1) *C is very fat if and only if $\sum_{k=1}^{\infty}(n_ka_k)^p<\infty$ for all $0<p<1$.*
(2) *C is minimally fat if and only if $\sum_{k=1}^{\infty}(n_ka_k)^p=\infty$ for all $0<p<1$.*
(3) *C is fairly fat if and only if there are constants $0<p<q<1$ such that $\sum_{k=1}^{\infty}(n_ka_k)^q<\infty$ and $\sum_{k=1}^{\infty}(n_ka_k)^p=\infty$.*
If C is thin, then
(4) *C is very thin if and only if $\sum_{k=1}^{\infty}(n_ka_k)^p=\infty$ for all $p>1$.*
(5) *C is minimally thin if and only if $\sum_{k=1}^{\infty}(n_ka_k)^p<\infty$ for all $p>1$.*
(6) *C is fairly thin if and only if there are constants $1<p<q<\infty$ such that $\sum_{k=1}^{\infty}(n_ka_k)^q<\infty$ and $\sum_{k=1}^{\infty}(n_ka_k)^p=\infty$.*

Remark 2. Theorem 3 extends the results in [3,7] since the uniform Cantor set in [3,7] satisfies the conditions of Theorem 3.