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The $p\lambda n$ fractal decomposition: Nontrivial partitions of conserved physical quantities

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ABSTRACT

A mathematical method for constructing fractal curves and surfaces, termed the $p\lambda n$ fractal decomposition, is presented. It allows any function to be split into a finite set of fractal discontinuous functions whose sum is equal everywhere to the original function. Thus, the method is specially suited for constructing families of fractal objects arising from a conserved physical quantity, the decomposition yielding an exact partition of the quantity in question. Most prominent classes of examples are provided by Hamiltonians and partition functions of statistical ensembles: By using this method, any such function can be decomposed in the ordinary sum of a specified number of terms (generally fractal functions), the decomposition being both exact and valid everywhere on the domain of the function.

1. Introduction

A fractal is a geometrical structure with non-integer dimension and the property of self-similarity [1–4]. Fractals abound in present day applications and are being used to describe a variety of phenomena in nature, ranging from nanoclusters [5] and semiconductors [6] to galaxies [7]. Fractals are important in the statistical mechanics of complex systems [8–10] and quantum mechanics [11–15]. Mathematically, they can arise from a variety of processes: iterated function systems [2,4], strange attractors of chaotic dynamical systems [16], critical phenomena [17], cellular automata [18–21], substitution systems [22], bitwise arithmetic [23], and arithmetic sequences [24]. Recently, a construction has been presented in terms of the hierarchical structure of a coin division problem [25,26].

In this article we present a new construction of a wide variety of fractal structures motivated by a general mathematical problem of interest to Physics that is easy to state and for which we provide a solution: *To find whether if given an arbitrary Hamiltonian H is it possible to split it into the ordinary sum of a given finite number $\lambda \in \mathbb{N}$ of (generally discontinuous) functions, bringing the Hamiltonian to the following form*

$$H = \sum_{n=0}^{\lambda-1} H_n \tag{1}$$

We call this problem the *partitioning* of the Hamiltonian. With this problem we exclude, of course, the trivial partitioning that results from dividing the Hamiltonian H by λ or by forming any linear combination of rational functions of H. We ask whether is it *always* possible to find λ *nontrivial* partitions of H *regardless* of the form of H (of course, the partitions H_n should themselves depend on H in a nontrivial way). Since we strive to give a general, constructive answer, we must renounce to demand smoothness (continuity and/or differentiability) of the parts H_n.

The answer to this question is, surprisingly, affirmative and we provide an explicit, general method, to construct each part H_n given H and λ. The partition is valid and exact everywhere on the energy surface and, by construction, the H_n's generally possess fractal features. Our approach makes use of a digit function that we have recently introduced in a new formulation of classical and quantum mechanics [27]

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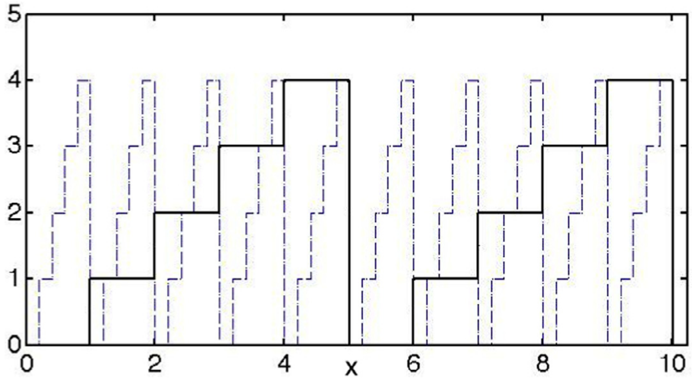

Fig. 1. The digit function $\mathbf{d}_5(0, x)$ (continuous curve) and $\mathbf{d}_5(-1, x)$ (dashed curve).

as well as elementary facts of abstract algebra (finite group theory) [28]. In the following section we first introduce the digit function elucidating some of its mathematical properties that we shall then use. We explicitly prove that the digit function induces a cyclic group structure on a finite set of integers. We then proceed to establish our main result, which is contained in what we term the $p\lambda n$ fractal decomposition: We prove that such a decomposition satisfies Eq. (1) thus solving the above problem affirmatively. We then extend these results to complex-valued quantities and prove a theorem that generalizes the method, significantly extending its domain of applicability. We finish with a short discussion of the mathematical results, sketching their connection to statistical mechanics and showing how they constitute a number-theoretic formulation of the latter.

2. The digit function and its basic properties

Let $p > 1$ be a natural number. Every real number x can be expanded in powers of p as

$$x = a_N p^N + a_{N-1} p^{N-1} + \cdots + a_0 + a_{-1} p^{-1} + a_{-2} p^{-2} + \cdots \tag{2}$$

where the a_k's are the so-called digits of the expansion and are either all positive or all negative, with absolute values being integers in the interval $[0, p-1]$. Eq. (2) is the representation of the real number x in base p. We shall call p the *radix* to avoid confusion with other uses of the word 'base' in physics. Eq. (2) can thus be written as

$$x = \text{sign}(x) \sum_{k=-\infty}^{\lfloor \log_p |x| \rfloor} |a_k| p^k \equiv \text{sign}(x) \sum_{k=-\infty}^{\lfloor \log_p |x| \rfloor} \mathbf{d}_p(k, |x|) p^k \tag{3}$$

where the upper bound in the sum gives the total number N of digits in the integer part of x as $N = \lfloor \log_p |x| \rfloor + 1$. Above, the digit function $\mathbf{d}_p(k, |x|) \equiv |a_k|$ of the real number x has been defined through the coefficients $|a_k|$. The digit function is explicitly given by [27]

$$\mathbf{d}_p(k, x) = \left\lfloor \frac{x}{p^k} \right\rfloor - p \left\lfloor \frac{x}{p^{k+1}} \right\rfloor \tag{4}$$

where $\lfloor \cdots \rfloor$ denote the closest lower integer (floor function) [29,30]. To see this note that, from Eq. (2) (we consider x nonnegative for simplicity)

$$\left\lfloor \frac{x}{p^k} \right\rfloor = a_N p^{N-k} + a_{N-1} p^{N-k-1} + \cdots + a_{k+1} p + a_k \tag{5}$$

From Eq. (2) we have, as well,

$$p \left\lfloor \frac{x}{p^{k+1}} \right\rfloor = a_N p^{N-k} + a_{N-1} p^{N-k-1} + \cdots + a_{k+1} p \tag{6}$$

whence, by subtracting both equations, we obtain Eq. (4).

The digit function $\mathbf{d}_p(k, x)$ is plotted in Fig. 1 for $p = 5$ and $k = 0$ (continuous curve) and $k = -1$ (dashed curve). It is a staircase of p levels taking discrete integer values between 0 and $p-1$. Each time that x is divisible by p^{k+1} the ascent of the staircase is broken and the level is set again to zero and a new staircase begins. We note that, in Eq. (3) we have, asymptotically

$$\lim_{p\to\infty} x = \lim_{p\to\infty} \left[\text{sign}(x) \sum_{k=-\infty}^{\lfloor \log_p |x| \rfloor} \mathbf{d}_p(k, |x|) p^k \right] \sim \text{sign}(x) \mathbf{d}_p(\lfloor \log_p |x| \rfloor, |x|) p^{\lfloor \log_p |x| \rfloor} \tag{7}$$

or, if the limit is strictly taken

$$\lim_{p\to\infty} x = \text{sign}(x) \mathbf{d}_\infty(0, |x|) \tag{8}$$

We note that in [27] x was taken nonnegative throughout (and hence $|x| = x$ and $\text{sign}(x) = 1$ there) and that the digit function as defined $\mathbf{d}_p(k, |x|)$ in [27] corresponds to $\mathbf{d}_p(k-1, |x|)$ here. We make here this notational adjustment to stay more in tune with standard mathematical textbooks [31] where the radix p expansion is discussed. Those textbooks remain at the level of introducing the coefficients a_k's as in Eq. (2) but do not discuss the associated digit function Eq. (4), whose properties we are about to exploit here.

We first note that the digit function allows Cantor's proof of the fact that *the real numbers cannot be counted with the natural numbers* to be presented in a concise way. If we concentrate on the real numbers in the unit interval $[0, 1)$ one can show that, already these, cannot be counted with the natural numbers. For let us assume an infinite list of such numbers $r_1, r_2, \ldots, r_n, \ldots$ where $n \in \mathbb{N}$. Then, the digit expansion