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The Poisson aggregation process

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ABSTRACT

In this paper we introduce and analyze the *Poisson Aggregation Process* (PAP): a stochastic model in which a random collection of random balls is stacked over a general metric space. The scattering of the balls' centers follows a general Poisson process over the metric space, and the balls' radii are independent and identically distributed random variables governed by a general distribution. For each point of the metric space, the PAP counts the number of balls that are stacked over it. The PAP model is a highly versatile spatial counterpart of the temporal $M/G/\infty$ model in queueing theory. The surface of the moon, scarred by circular meteor-impact craters, exemplifies the PAP model in two dimensions: the PAP counts the number of meteor-impacts that any given moon-surface point sustained. A comprehensive analysis of the PAP is presented, and the closed-form results established include: general statistics, stationary statistics, short-range and long-range dependencies, a Central Limit Theorem, an Extreme Limit Theorem, and fractality.

1. Introduction

In queueing theory – the branch of mathematics that studies the statistical behavior of queues – "$M/G/\infty$" is the common acronym for a fundamental model of systems characterized by a Markovian arrival process of customers (represented by the letter "M"), a general service time of the incoming customers (represented by the letter "G"), and an infinite battery of servers poised to attend the incoming customers (represented by the symbol "∞") [1–5]. From a physical perspective the $M/G/\infty$ model can be described as follows: particles enter a system of interest according to a Poisson stream of arrivals; each particle, independently of everything else, stays in the system for a random duration of time; the particles' sojourn times in the system share a common statistical distribution. Evidently, the number of particles in the system fluctuates randomly, and the stochastic process tracking this number is termed the $M/G/\infty$ process. The origins of the $M/G/\infty$ model stem from particle physics – where this model arose in the context of Geiger–Muller counters; the application of the $M/G/\infty$ model in the context of queueing theory came only after its 'physics debut' [6–12]. Interestingly, the $M/G/\infty$ model and its variants were recently applied in the biophysics of gene expression [13]; more generally, we note that in the recent years queueing models are attracting quite an interest in physics, e.g., [14–18].

The goal of this paper is to introduce and analyze a general spatial counterpart of the temporal $M/G/\infty$ process: the *Poisson Aggregation Process* (PAP). Pictorially, the $M/G/\infty$ process can be constructed by stacking a random collection of random intervals over the timeline – each interval representing the sojourn time of a particle arriving to the system. With this picture in mind, the conceptual transition from a temporal setting to a general spatial setting is rather straightforward: stack a random collection of random sets over a space of interest. To make this transition precise, three model-foundations must be specified: the underlying space, the random scattering of the sets, and the geometry and statistics

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of the sets. In the PAP model these foundations are as follows:

Space. The underlying space is taken to be a general *metric space* [19–21]. Metric spaces accommodate a wide variety of settings including Euclidean spaces and Euclidean domains of arbitrary dimension, non-Euclidean spaces such as elliptic or hyperbolic spaces, general surfaces and landscapes, general fractal objects, general networks, and more.

Scattering. The scattering of the sets is considered to be according to a general *Poisson process* over the underlying metric space [4,22–24]. Poisson processes are a potent statistical methodology for the random scattering of points over general spaces, with a broad span of applications ranging from the abovementioned queueing theory [25] to insurance and finance [26], and from fractal objects [27] to power-laws [28].

Sets. The sets are taken to be the simplest generalization of one-dimensional intervals – balls in the metric space. The ball centers are the points of the aforementioned Poisson process, and analogously to the $M/G/\infty$ model: each ball, independently of everything else, has a random radius, and the balls' radii share a common statistical distribution.

A vivid and tangible two-dimensional example of the PAP model is the surface of our moon – scarred with numerous circular craters caused by meteor impacts. In this example the metric space is the moon surface, the randomly scattered points are meteor-impact epicenters, and the balls are the circular impact craters. The distribution of the craters radii is induced by the distribution of the meteor sizes.

On the one hand, the underlying metric space and the Poisson-process scattering allow for great model-versatility; on the other hand, this choice of model-foundations turns out to be amenable to mathematical analysis, and yields closed-form analytic results. Thus, these model-foundations attain a fine balance between generality and tractability. The exposition and the analysis of the PAP model are as follows:

The paper begins with a detailed description and construction of the PAP model (Section 2), followed by a general statistical analysis (Section 3), and by a general stationarity analysis (Section 4) – which, in turn, is followed by a discussion (Section 5). Then, two stochastic limit laws of the PAP model are established – a Central Limit Theorem yielding a universal Gaussian approximation (Section 6), and an Extreme Limit Theorem yielding a universal 'extreme-value' limit with fractal features (Section 7). The paper ends with an illustrative example demonstrating the application of the general PAP results (Section 8), and with a conclusion (Section 9). The derivations of the various PAP results are detailed in the Methods section (Section 10).

Short glossary of notation that shall be used along the paper

Indicator functions: $\mathbf{I}\{A\}$ is the indicator function of an event A; namely, $\mathbf{I}\{A\}=1$ if the event A occurred, and $\mathbf{I}\{A\}=0$ if the event A did not occur.

Poisson distribution: $\pi_n(\mu)=\exp(-\mu)\mu^n/n!$ is the probability that a Poisson-distributed random variable, with mean μ ($\mu>0$), yields the outcome n ($n=0,1,2,\ldots$).

Expectation: $\mathbf{E}[\,\cdot\,]$ is the operation of mathematical expectation; namely, $\mathbf{E}[\xi]$ is the mean of the real-valued random variable ξ.

Asymptotic equivalence: $\approx$ denotes asymptotic equivalence of functions; specifically, if $f_1(t)$ and $f_2(t)$ are functions defined on the non-negative half-line ($t\geq 0$), then $f_1(t)\approx f_2(t)$ means that $\lim_{t\to\infty}[f_1(t)/f_2(t)]=\omega$, where ω is a positive limit.

2. The PAP model

In a nutshell, the Poisson Aggregation Process (PAP) stacks a random collection of random balls over a given metric space $\mathcal{S}$. As noted in the Introduction, the metric space can be an Euclidean space or an Euclidean domain of arbitrary dimension, a non-Euclidean space such as an elliptic space or a hyperbolic space, a surface or a landscape, a fractal object, a general network, etc. The precise definition of the PAP is described as follows:

The geometry of the underlying metric space $\mathcal{S}$ is characterized by its distance function $d(x,y)$, which represents the distance between the points $x,y\in\mathcal{S}$ [19–21]. The stacking is by a countable collection of balls labeled by the index i – with the point $P_i\in\mathcal{S}$ representing the center of ball i, and with the positive number R_i representing the radius of ball i. Specifically, ball i encompasses all the points of $\mathcal{S}$ that are within a distance R_i from its center point P_i, and is thus given by the set $\{x\in\mathcal{S}|d(x,P_i)<R_i\}$. As noted above the balls are random, and their statistics are considered to be as follows:

- The collection of the ball-centers $\mathcal{P}=\{P_i\}$ forms a general *Poisson process* over the space $\mathcal{S}$.
- The ball-radii $\{R_i\}$ are *independent and identically distributed*copies of a general positive-valued random variable R – henceforth termed the 'generic radius'.
- The collection of the ball-radii $\{R_i\}$ is *independent* of the collection of the ball-centers $\mathcal{P}=\{P_i\}$.

Poisson processes are a highly versatile and widely applicable statistical methodology to model the random scattering of points over general spaces and domains [4,22–24]. The distribution of the Poisson process $\mathcal{P}$ is governed by its intensity – a measure $\lambda(ds)$ over the space $\mathcal{S}$ – according to the two following rules: (i) the number of ball-centers residing in the measurable subset $S\subset\mathcal{S}$ is a Poisson-distributed random variable with mean $\lambda(S)=\int_S\lambda(ds)$; (ii) the numbers of ball-centers residing in disjoint subsets of $\mathcal{S}$ are independent random variables. Also, we denote by $\Phi(r)=\Pr(R\leq r)$ ($r\geq 0$) the cumulative distribution function of the generic radius R, and by $\bar{\Phi}(r)=1-\Phi(r)=\Pr(R>r)$ ($r\geq 0$) the corresponding tail function.

The PAP is defined as a collection of integer-valued random variables indexed by the points of the metric space: $\{N(x);x\in\mathcal{S}\}$, with

$$N(x)=\sum_i \mathbf{I}\{d(x,P_i)<R_i\}\,. \tag{1}$$

Namely, the random variable $N(x)$ counts the number of balls that are stacked over the point x, and it attains the integer values $\{0,1,2,\ldots\}$. For example, one may envision a shower of meteors impacting a given surface $\mathcal{S}$, say the surface of our moon. The meteors are labeled by the index i, the point P_i is the impact-epicenter of meteor i, and R_i is the impact-range of meteor i . In this example the random variable $N(x)$ counts the number of meteor impacts that the point x sustained.