Chaos, Solitons and Fractals 83 (2016) 54–66

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

# Role of ergodicity in the transient Fluctuation Relation and a new relation for a dissipative non-chaotic map

Paolo A. Adamo [a], Matteo Colangeli [b,*], Lamberto Rondoni [a,c]

[a] Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
[b] Gran Sasso Science Institute, Viale F. Crispi 7, 00167 L' Aquila, Italy
[c] INFN, Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy

ARTICLE INFO

*Article history:*
Received 22 April 2015
Accepted 19 November 2015
Available online 11 December 2015

*Keywords:*
Nonequilibrium
Fluctuation Relations
Ergodicity
Dissipation

ABSTRACT

Toy model dynamical systems, such as the baker maps, are useful to shed light on some of the conditions verified by deterministic models in non-equilibrium statistical physics. We investigate a 2D dynamical system, enjoying a weak form of reversibility, with peculiar basins of attraction and steady states. In particular, we test the conditions required for the validity of the transient Fluctuation Relation. Our analysis illustrates by means of concrete examples why ergodicity of the equilibrium dynamics (also known as "ergodic consistency") seems to be a necessary condition for the transient Fluctuation Relation. This investigation then leads to the numerical verification of a kind of transient relation which, differently from the usual transient Fluctuation Relation, holds only asymptotically. At the same time, this relation is not a steady state Fluctuation Relation, because the steady state is a fixed point without fluctuations.



## 1. Introduction

The theory of Fluctuation Relations (FR) originated with the seminal work of Evans et al. [1], and motivated many works, beginning with [2] for the transient FR and [3] for the steady state FR. These relations became increasingly popular in statistical mechanics, as they describe statistical properties of systems far from equilibrium [4–7]. Therefore it is important to clarify the assumptions under which such relations hold. The vast majority of works on the derivation of the steady state FR for the phase space contraction rate $\Lambda$ is based on the assumption that the invariant probability measure is smooth along the unstable direction, and that the dynamics is time reversal invariant [8,9]. The purpose of time reversibility in those works is to associate each trajectory segment of given length $n$ and phase space contraction $\Lambda_n$ with a counterpart yielding opposite phase space contraction $-\Lambda_n$, so that the ratio of the probabilities of positive and negative phase space contractions can be compared. However, while the condition of time reversal invariance is sufficient for every trajectory segment to have such a counterpart, this condition is not always verified by models of physical interest and, like other ingredients of the original proofs, it is not necessary for the FR to hold, see e.g. Refs. [10,11]. In general, the identification of the minimal requirements for the steady state and for the transient FR to hold remains an open question, although it is important in order to delimit the range of applicability of such relations. Consequently, numerous papers have been devoted to that, consider e.g. Refs. [12,13] for stochastic systems, Ref. [14] for billiards, Refs. [10,11] for simple maps, and Ref. [8] for granular fluids. Similar investigations on the applicability of mathematical relations of physical interest are common, because derivations usually involve mathematically sufficient ingredients which may not be necessary, or which may be physically unlikely, thus hiding

* Corresponding author. Tel.: +39 0862 4280 412.
*E-mail addresses:* adamatrice@gmail.com (P.A. Adamo), matteo.colangeli@gssi.infn.it (M. Colangeli), lamberto.rondoni@polito.it (L. Rondoni).

http://dx.doi.org/10.1016/j.chaos.2015.11.025

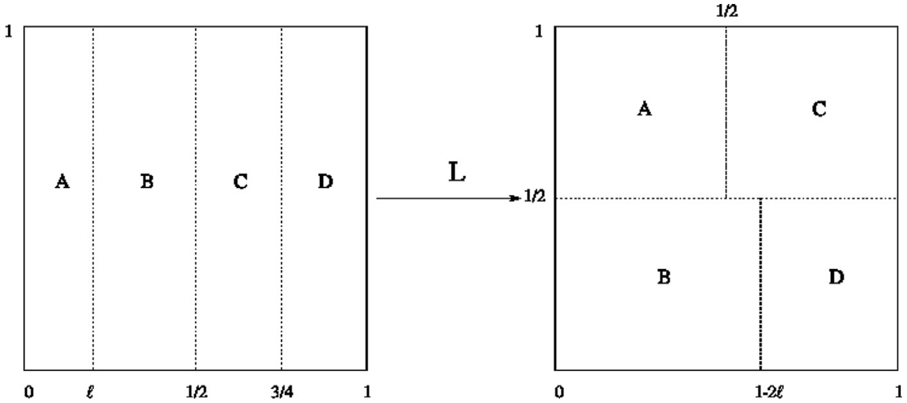


**Fig. 1.** Map $L$ defined as the composition of the maps described by the Eqs. (1) and (5).

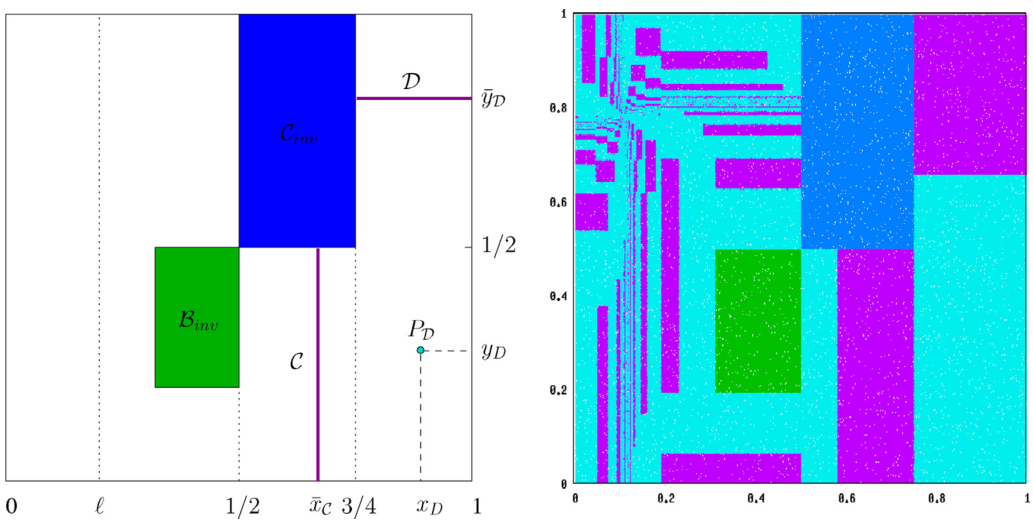


**Fig. 2.** *Left panel*: the attractors of the map $L$. *Right panel*: the corresponding basins of attraction (shown is the case with $\ell = 0.19$). Points in the area coloured in turquoise will converge in the steady state to the attractor $P_{\mathcal{D}}$, while the points lying in the purple regions will collapse to the two orthogonal lines referred to, in the left panel, as $\mathcal{C}$ and $\mathcal{D}$. Those orbits are only possible if the parameter $\ell$ satisfies $\frac{1}{8} \le \ell \le \frac{1}{4}$. Finally the green and blue regions are invariant regions which remain unchanged under the dynamics, and coincide with the regions $B_{inv}$ and $C_{inv}$ portrayed on the left panel. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the physical mechanisms at work; cf. Refs. [15,16] for the ergodic hypothesis, Refs. [17,18] for diffusion [1], and Refs. [22–26] for linear response. Also, the predictive value of a theory is reduced, if the relevant range of applicability is not identified.

Toy models have proved to be an invaluable source of insight in this respect and, in particular, simple one-dimensional and two-dimensional dynamical systems have been used in non-equilibrium statistical physics, because amenable to thorough analysis [27,28]. Obviously, their dynamics cannot be taken to realistically represent the complexity of many interacting particles, but they efficiently serve as benchmarks for various assumptions; e.g. one counterexample suffices to show that a given theory needs to more precisely outlined. For instance, Colangeli et al. [10] prove the validity of the steady state FR for the phase space contraction rate of a hyperbolic dynamical system whose invariant measure is discontinuous along the unstable direction. Colangeli and Rondoni [11] obtain the validity of the steady state FR for a dynamical system which is not time reversal invariant but it satisfies a weaker symmetry, analogous to that used for stochastic FR, which allows paths with opposite phase space contraction to be paired.

In the present paper we take one step further, considering the transient FR for a map $L$ of the unit square $\mathcal{U} = [0, 1] \times [0, 1]$, depending on one parameter $\ell$ which can be tuned to produce either non-dissipative (hereafter called "equilibrium") dynamics, or dissipative dynamics (in which case we speak of "non-equilibrium" dynamics). In this framework, non-dissipative means that volumes of $\mathcal{U}$ are preserved in time, analogously to Hamiltonian dynamics, while dissipa-

[1] It is worth noticing that stochastic processes for nonequilibrium extended systems enjoying hydrodynamic limits and diffusion also obey the steady state FR. This is due to their Markovian microscopic structure, see e.g. Refs. [19–21].