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Suppression of periodic structures and the onset of hyperchaos in a parameter-space of the Baier–Sahle flow

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ABSTRACT

We investigate a two-dimensional parameter-space of the Baier–Sahle flow, which is a mathematical model consisting of a set of n autonomous, four-parameter, first-order nonlinear ordinary differential equations. By using the Lyapunov exponents spectrum to numerically characterize the dynamics of the model in the chosen parameter-space, we show that for $n = 3$ it presents typical periodic structures embedded in a chaotic region, forming a spiral structure that coils up around a focal point while period-adding bifurcations take place. We also show that these structures are destroyed as n is increased, as well as we delimit hyperchaotic regions with two or more positive Lyapunov exponents in the investigated parameter-space, for n greater than 3.

1. Introduction

The object of investigation in this work was proposed by Baier and Sahle [1]. It is an n-dimensional mathematical model not directly associated with any system of the Nature, given by

$$\begin{aligned} \dot{x}_1 &= -x_2 + ax_1, \\ \dot{x}_m &= x_{m-1} - x_{m+1}, \\ \dot{x}_n &= c + bx_n(x_{n-1} - d), \end{aligned} \tag{1}$$

where $n \geq 3$, $m = 2, \ldots, n-1$, and $a > 0$, b, c, d are real parameters. Although system (1) is a mathematical model not directly associated with any system of the real world, therefore an abstract system, there are chemical reactions that can be modeled with success by sets of first-order ordinary differential equations which works analogously to the Baier–Sahle flow, as shown in Ref. [1]. This is certainly an indicator of its importance. It is noteworthy that system (1) presents only one quadratic nonlinearity, namely in the $\dot{x}_n$ equation, and that hyperchaos is possible to be achieved for $n \geq 4$. System (1) was originally proposed in Ref. [1] as a design strategy to create chaotic attractors with an increasing number of positive Lyapunov exponents, as the dimension of the system, n, is increased. The Lyapunov exponents spectrum was numerically computed in Ref. [1], for different fixed sets of parameters (a, b, c, d) and dimension n. For $n = 4$, $a = 0.4$, $b = 4.0$, $c = 0.1$, $d = 2.0$ were found two positive Lyapunov exponents. For $n = 5$, $a = 0.33$, $b = 4.0$, $c = 0.1$, $d = 2.0$ were found three positive Lyapunov exponents. For $n = 7$, $a = 0.32$, $b = 4.0$, $c = 0.1$, $d = 2.0$ were found four positive Lyapunov exponents, therefore one less than the possible maximum number. For $n = 9$, $a = 0.30$, $b = 4.0$, $c = 0.1$, $d = 2.0$ were found six positive Lyapunov exponents, again one less than the possible maximum number.

More recently the method of competitive modes [2], which is used in order to verify the necessary conditions for a dynamical system to display chaos, was used to investigate

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the n-dimensional Baier–Sahle flow [3]. The reason for the choice of this model is due to its simplicity, which makes it analytically tractable. In addition to analytical results, Ref. [3] provides explicit numerical results for $n = 4$ and $n = 5$, with $a = 0.4$, $b = 4.0$, $c = 0.1$, and $d = 2.0$. Synchronization of two identical n-dimensional hyperchaotic Baier–Sahle systems was investigated by Nik and co-workers [4], that also investigated hyperchaos control via an adaptive feedback control scheme. In addition to an analytical analysis, Ref. [4] also provides numerical simulations for $n = 4$ and $n = 5$. The set of parameters was kept fixed respectively as $a = 0.4$, $b = 4.0$, $c = 0.1$, $d = 2.0$ and $a = 0.33$, $b = 4.0$, $c = 0.1$, $d = 2.0$. As far as we know, investigations considering the Baier–Sahle flow conducted until now, consider all the parameters fixed either the variation of only one of them, namely a.

In the present work we consider always the simultaneous variation of two parameters. More specifically, in this work we investigate the (a, b) two-dimensional parameter-space of the Baier–Sahle model for different n values, namely 3, 4, 5, and 6, with c and d kept fixed respectively as 0.1 and 2.0. The paper is organized as follows. In Section 2 (a, b) parameter-spaces of the three-dimensional Baier–Sahle flow are presented and interpreted. Sections 3, 4, and 5 consider (a, b) parameter-spaces related respectively to the four-, five-, and six-dimensional cases, for which hyperchaos is a possible dynamical behavior. The paper is finalized in Section 6.

2. The Baier–Sahle flow for $n = 3$

Let us consider the Baier–Sahle flow model (1) for $n = 3$, which is given by

$$\begin{aligned} \dot{x}_1 &= -x_2 + a x_1, \\ \dot{x}_2 &= x_1 - x_3, \\ \dot{x}_3 &= c + b x_3 (x_2 - d), \end{aligned} \tag{2}$$

and which has only one nonlinearity, namely quadratic in the $\dot{x}_3$ equation. Before continuing, we would like to point out that the dynamics of three-dimensional mathematical models having quadratic nonlinearities has been the subject of very recent investigations [5–7].

Fig. 1 shows the (a, b) parameter-space for the Baier–Sahle flow model (2), obtained by plotting the largest Lyapunov exponent (LLE) on a $10^3 \times 10^3$ grid of equally spaced points. The remaining parameters, c and d, were kept fixed respectively as 0.1 and 2.0. System (2) was integrated using a fourth order Runge–Kutta algorithm with a fixed time step equal to 10^{-3}, and considering 2×10^6 steps to compute the average involved in the computation of each of the 1×10^6 LLE. Before each computation, we are ruling out the first 5×10^5 steps, to disregard the transient behavior. The initial condition for each one of these 1×10^6 trajectories in the phase-space, was always fixed as $(x_1, x_2, x_3) = (0.1, 0.3, 0.5)$.

Our understanding regarding the interpretation of the value of the LLE (positive, negative, or zero) is based on the work by Wolf and co-workers [8], although we know that a negative LLE does not indicate, in general, stability, as well as a positive LLE does not indicate, in general instability [9]. Hence, a negative LLE characterizes the related trajectory in the phase-space as equilibrium point, a zero LLE as periodic or quasiperiodic, and a positive LLE as chaotic. Colors in Fig. 1

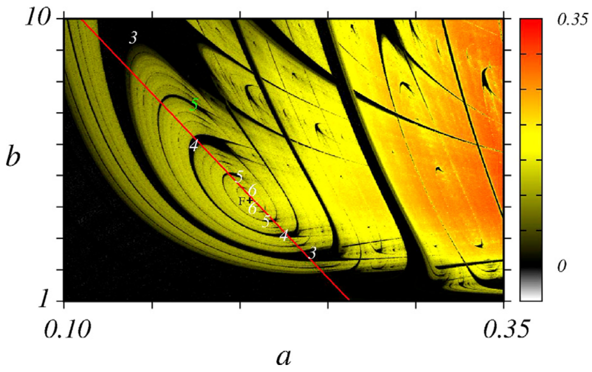

Fig. 1. The (a, b) parameter-space for system (2) showing some spirals. Color is related to the magnitude of the largest Lyapunov exponent. Number is related to the period of the respective structure (see text). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

are related to the magnitude of the respective LLE. A positive LLE is indicated by a continuously changing yellow to red color, a negative LLE by a continuously changing white to black, and the black color itself indicates a zero LLE, as the scale shown in the column at right side in Fig. 1.

An interesting feature displayed by the parameter-space diagram in Fig. 1, is the presence of typical periodic structures in black, embedded in a chaotic region in yellow–red, and showing some type of organization. An example is the set arranged along the red straight line $b = -58.7809\,a + 16.4387$, where can be seen periodic structures organized in spirals, with a focal point F numerically estimated at $(a, b) = (0.2012, 4.6402)$. In the scale of Fig. 1, the most outstanding of these spirals is that in which some few elements are distinguished by a white number representing the period of itself. Note that in a single spiral, each structure is connected to the other structure by *legs* of periodicity that originate from the structures itself. In the spiral detached in Fig. 1, the main periodicity of the structures increases by an amount equal to 1 as they coil up around the focal point F. This means that in the considered spiral, the periodic structures are organized in a period-adding bifurcation cascade that increases the periodicity by 1, as the spiral converges to the focal point F.

With regard specifically to the most detached spiral in Fig. 1, if we start to move along the *leg* joined to the period-3 structure at right, in the lower portion of the parameter-space, we arrive at the period-3 structure at left in the upper portion. Continuing the moving, now from this period-3 structure, along the *leg* joined to it, we arrive at the period-4 structure at right in the lower portion. This behavior is recurrent, and the result is the $3 \to 3 \to 4 \to 4 \to 5 \to 5 \to 6 \to 6 \to \cdots$ periodic sequence visible in Fig. 1, as we move along the *legs* closer and closer to the focal point F. Similar behavior occurs with other spirals, for instance that spiral not so visible in the scale used, which includes the period-5 structure numbered in green in Fig. 1.

It is important to note that these spiral bifurcations were observed before in parameter-spaces of electronic circuits [10–12], in a Rössler model [13], in a chemical oscillator [10], in a Hopfield neural network [14], in modified optical injection semiconductor lasers [15], and in a tumor growth