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On the limit cycles of perturbed discontinuous planar systems with 4 switching lines

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ABSTRACT

Limit cycle bifurcations for a class of perturbed planar piecewise smooth systems with 4 switching lines are investigated. The expressions of the first order Melnikov function are established when the unperturbed system has a compound global center, a compound homoclinic loop, a compound 2-polycycle, a compound 3-polycycle or a compound 4-polycycle, respectively. Using Melnikov's method, we obtain lower bounds of the maximal number of limit cycles for the above five different cases. Further, we derive upper bounds of the number of limit cycles for the later four different cases. Finally, we give a numerical example to verify the theory results.

1. Introduction

Piecewise smooth (PWS) dynamical systems are now commonplace used as models in many fields, such as control theory, electronics, biology, medicine, mechanical systems with dry frictions, information and economics, see e.g. [1,10–13,17,22] and the references therein. As a consequence, rich works from different perspectives have been done on PWS systems, see e.g. [2–9,14,16,18–20], which promote the development of bifurcation theory rapidly. Just like smooth systems, one of the most significant problem about PWS systems is how to determine the number of limit cycles, though it is much more difficult for them. So far, there are at least three general ways to engender multiple limit cycles for a given PWS system, which are to study the small amplitude limit cycles through Hopf bifurcations or center bifurcations [6], to study the large amplitude limit cycles bifurcating from a periodic annulus [3], and to study the multiple limit cycles through homoclinic bifurcations [4], respectively.

Most of the aforementioned works about PWS systems are assumed that the phase space is divided into two disjoint connected subregions by one switching line, see [2–6,8,10,12–14,18–20]. For example, in [4], Liang et al. studied the bifurcation of limit cycles from homoclinic loops by perturbing a piecewise Hamiltonian system with one switching line $x = 0$. Han and Zhang in [6] considered Hopf bifurcations in non-smooth planar systems with one switching line $x = 0$.

However, as pointed out in [16], discontinuities may occur on multiple lines or even on nonlinear curves or surfaces. Recently, a few works have been done on PWS systems with multiple switching lines, see [7,9,16,17]. In the literature [7], Hu and Du discussed the bifurcation of periodic orbits in the following perturbed planar discontinuous system with discontinuities on the switching lines $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$:

$$\dot{x} = g_k(x) + \epsilon f_k(x), \quad x \in D_k, \quad k = 1, 2, \ldots, m, \tag{1.1}$$

and the unperturbed system is given by

$$\dot{x} = g_k(x), \quad x \in D_k, \quad k = 1, 2, \ldots, m, \tag{1.2}$$

where $g_k, \ f_k \in C^2(D_k \cup \Sigma_k \cup \Sigma_{k+1}, R^2)$, $\Sigma_k = \{x \in \mathbb{R}^2 : c_k^T x = 0\}$ with $c_k \in \mathbb{R}^2$ satisfying $\| c_k \| = 1$, D_k is the open

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region between Σ_k and Σ_{k+1} for $k = 1, 2, \ldots, m$, and $\Sigma_{m+1} = \Sigma_1$, $|\epsilon| \le \epsilon_0$ for some $\epsilon_0 > 0$.

Mainly motivated by authors in [4,7], in this paper, we study the limit cycle bifurcations for a class of perturbed planar discontinuous systems with 4 switching lines by the Melnikov method [15,21]. We suppose that for the unperturbed system one of the following five different cases occurs: (i) a compound global center; (ii) a compound homoclinic loop; (iii) a compound 2-polycycle; (iv) a compound 3-polycycle; (v) a compound 4-polycycle. We first establish the expressions of the first order Melnikov function $M(h)$ for the above mentioned five cases. Then using the expressions we obtain lower bounds of the maximal number of limit cycles in cases (i)–(v), and further derive upper bounds of the number of limit cycles in cases (ii)–(v).

The layout of the paper is as follows. In Section 2, some notations, basic assumptions and a fundamental lemma are presented. In Sections 3–7, we give a general expression of $M(h)$ and obtain the number of limit cycles in cases (i)–(v), respectively. In Section 8, a numerical example is given to verify the theory results.

2. Basic assumptions and a fundamental lemma

We assume that the state space $\mathbb{R}^2$ is spilt into 4 disjoint regions by the four rays $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$, where

$$\begin{aligned}\Sigma_1 &= \{(x,y)|y=0,\ x\ge 0\}\\ &= \{(x,y)|c_1^T(x,y)^T=0,\ x\ge 0\},\quad c_1^T=(0,1),\\ \Sigma_2 &= \{(x,y)|x=0,\ y\ge 0\}\\ &= \{(x,y)|c_2^T(x,y)^T=0,\ y\ge 0\},\quad c_2^T=(-1,0),\\ \Sigma_3 &= \{(x,y)|y=0,\ x\le 0\}\\ &= \{(x,y)|c_3^T(x,y)^T=0,\ x\le 0\},\quad c_3^T=(0,-1),\\ \Sigma_4 &= \{(x,y)|x=0,\ y\le 0\}\\ &= \{(x,y)|c_4^T(x,y)^T=0,\ y\le 0\},\quad c_4^T=(1,0).\end{aligned}$$

Let $\Sigma_5 = \Sigma_1$. Denote by D_k the open region between Σ_k and Σ_{k+1} for $k = 1, 2, 3, 4$, that is,

$$\begin{aligned}D_1 &= \{(x,y)|x>0,\ y>0\},\quad D_2=\{(x,y)|x<0,\ y>0\},\\ D_3 &= \{(x,y)|x<0,\ y<0\},\quad D_4=\{(x,y)|x>0,\ y<0\}.\end{aligned}$$

Obviously, $D_k \cap (\Sigma_k \cup \Sigma_{k+1}) (k = 1, 2, 3, 4)$ is empty.

Consider the following planar piecewise system defined on $D_1 \cup D_2 \cup D_3 \cup D_4$

$$\begin{aligned}\dot{x} &= H_{ky} + \epsilon p_k(x,y),\\ \dot{y} &= -H_{kx} + \epsilon q_k(x,y),\quad (x,y)\in D_k,\quad k=1,2,3,4,\end{aligned} \tag{2.1}$$

where $H_{ky} = \frac{\partial H_k(x,y)}{\partial y}$, $H_{kx} = \frac{\partial H_k(x,y)}{\partial x}$, and $H_k, p_k, q_k \in C^\infty(D_k \cup \Sigma_k \cup \Sigma_{k+1}, \mathbb{R}^2)$, $|\epsilon| \ge 0$ is small. When $\epsilon = 0$, the unperturbed system of (2.1) is given by

$$\dot{x} = H_{ky},\quad \dot{y} = -H_{kx},\quad (x,y)\in D_k,\quad k=1,2,3,4. \tag{2.2}$$

We first make the following two hypotheses for system (2.2): (H1) There exist four points $A_1 = (a_1(h), 0)$, $A_2 = (0, a_2(h)), A_3 = (a_3(h), 0), A_4 = (0, a_4(h))$ and an interval $J = (\alpha, \beta)$ satisfying

$H_1(A_2) = H_1(A_1) = h$, $H_2(A_3) = H_2(A_2)$, $H_3(A_4) = H_3(A_3), H_4(A_1) = H_4(A_4)$ for $h \in J$, where $a_1(h) \ne a_3(h), a_2(h) \ne a_4(h)$.

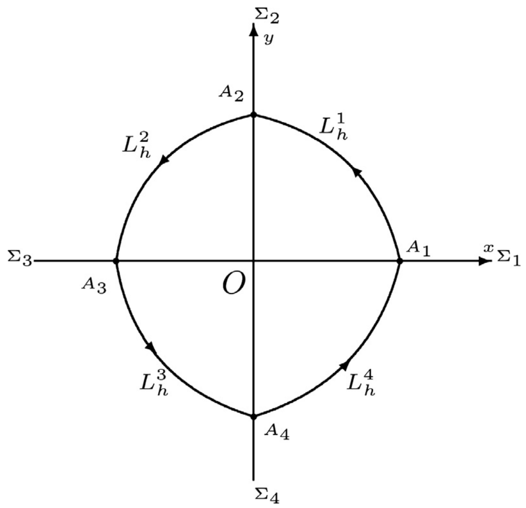

Fig. 2.1. The closed orbits of (2.2).

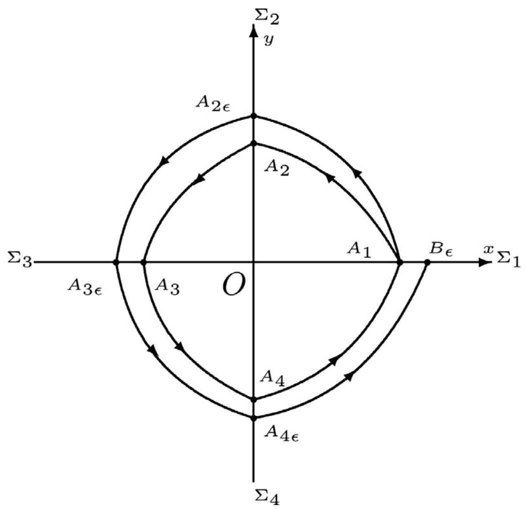

Fig. 2.2. The construction of the Poincaré map for each subsystem of (2.1).

(H2) For $h \in J$, system (2.2) has a family of periodic orbits $L_h = L_h^1 \cup L_h^2 \cup L_h^3 \cup L_h^4$ crossing $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ counterclockwise. For $k = 1, 2, 3, 4$, L_h intersects Σ_k at $A_k \in \Sigma_k$, and has an intersection L_h^k with D_k. That is,

$$\begin{aligned}L_h^1 &= \{(x,y)\in D_1 : H_1(x,y)=h\},\\ L_h^j &= \{(x,y)\in D_j : H_j(x,y)=H_j(A_j)\},\quad j=2,3,4.\end{aligned}$$

See Fig. 2.1.

Now we define a bifurcation function $F(h, \epsilon)$ of system (2.1), see Fig. 2.2. By (H2), the orbit of system (2.1) starting from A_1 at Σ_1 crosses $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ counterclockwise. Let $A_{j\epsilon}$ denote its first intersection point with Σ_j, $j = 2, 3, 4$, and B_ϵ denote the orbit's second intersection point with Σ_1 when it returns to Σ_1 for the first time. Set $A_{2\epsilon} = (0, a_{2\epsilon}(h))$, $A_{3\epsilon} = (a_{3\epsilon}(h), 0)$, $A_{4\epsilon} = (0, a_{4\epsilon}(h))$, $B_\epsilon = (a_{1\epsilon}(h), 0)$. Then by Theorem 2.2 in [7] we can define

$$H_4(B_\epsilon) - H_4(A_1) = \epsilon F(h, \epsilon). \tag{2.3}$$