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On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions

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ABSTRACT

We investigate a coupled system of fractional differential equations with nonlinearities depending on the unknown functions as well as their lower order fractional derivatives supplemented with coupled nonlocal and integral boundary conditions. We emphasize that the problem considered in the present setting is new and provides further insight into the study of nonlocal nonlinear coupled boundary value problems. We present two results in this paper: the first one dealing with the uniqueness of solutions for the given problem is established by applying contraction mapping principle, while the second one concerning the existence of solutions is obtained via Leray–Schauder's alternative. The main results are well illustrated with the aid of examples.

1. Introduction

In this paper we study a coupled system of nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), \quad y(t), \quad D^{\gamma}y(t)), \quad t \in [0,T], \\ \qquad 1 < \alpha \le 2, \quad 0 < \gamma < 1, \\ D^{\beta}y(t) = g\big(t, x(t), \quad D^{\delta}x(t), \quad y(t)\big), \quad t \in [0,T], \\ \qquad 1 < \beta \le 2, \quad 0 < \delta < 1, \end{cases} \tag{1.1}$$

supplemented with coupled nonlocal and integral boundary conditions of the form:

$$\begin{cases} x(0) = h(y), \quad \int_0^T y(s)ds = \mu_1 x(\eta), \\ y(0) = \phi(x), \quad \int_0^T x(s)ds = \mu_2 y(\xi), \quad \eta, \xi \in (0,T), \end{cases} \tag{1.2}$$

where ${}^cD^i, i = \alpha, \beta, \gamma, \delta$ denote the Caputo fractional derivatives of order i, $i = \alpha, \beta, \gamma, \delta$ respectively, $f, g : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h, \phi : C([0,T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and μ_1, μ_2 are real constants.

The study of boundary value problems for linear and nonlinear differential equations constitutes an important and popular field of research in view of occurrence of such problems in a variety of disciplines of pure and applied sciences. In recent years, fractional-order boundary value problems have been extensively investigated and a great deal of work ranging from theoretical development to applications can be found in the literature on the topic, for instance, see [1–14] and the references cited therein.

The importance of fractional calculus is now quite perceptible as the mathematical modeling of several real world phenomena via the tools of this branch of mathematics has led to exploration of new hereditary and memory characteristics of the processes and materials involved

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in the phenomena. In consequence, integer-order models in many physical and engineering disciplines such as viscoelasticity, biophysics, blood flow phenomena, chemical processes, control theory, wave propagation, signal and image processing, etc., have been transformed to their fractional-order counterparts. For further details, we refer the reader to the texts [15–18], while the facts about the recent history of fractional calculus can be found in [19].

The study of coupled systems of fractional-order differential equations is found to be of great value and interest in view of the occurrence of such systems in a variety of problems of applied nature. Examples include quantum evolution of complex systems [20], distributed-order dynamical systems [21], Chua circuit [22], Duffing system [23], Lorenz system [24], anomalous diffusion [25,26], systems of nonlocal thermoelasticity [27,28], secure communication and control processing [29], synchronization of coupled fractional-order chaotic systems [30–33], etc. Fractional differential systems are more suitable for describing the physical phenomena possessing memory and genetic characteristics.

Nonlocal conditions play a key role in describing some peculiarities of physical, chemical or other processes happening at various positions inside the domain, which is obviously not possible with the end-point (initial/boundary) conditions. For the historical background of these conditions, we refer the reader to the works [34–36].

Integral boundary conditions are found to be important and significant in the study of Computational fluid dynamics (CFD) studies related to blood flow problems. In the analysis of such problems, cross-section of blood vessels is assumed to be circular, which is not always justifiable. In order to cope this problem, integral boundary conditions provide an effective and applicable approach. More details can be found in [37]. Also, integral boundary conditions have useful applications in regularizing ill-posed parabolic backward problems in time partial differential equations, see for example, mathematical models for bacterial self-regularization [38].

Some recent investigations on coupled systems of fractional order differential equations, including nonlocal and integral boundary conditions, can be found in [39–46] and the references cited therein.

The paper is organized as follows. In Section 2, we recall some definitions from fractional calculus and present an auxiliary lemma. The main results for the coupled system of nonlinear fractional differential equations with coupled nonlocal and integral boundary conditions are obtained via contraction mapping principle and Leray–Schauder alternative, and are presented in Section 3.1. Since the methods of proofs employed in this paper are the standard ones in the contexts of fractional differential equations with boundary conditions, for instance, see [43], we omit some details in the proofs of our results. The paper concludes with illustrative examples.

2. Preliminaries

First of all, we recall definitions of fractional integral and derivative [15,16].

Definition 2.1. The Riemann–Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2. For at least n-times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Now we prove an auxiliary result which is pivotal to define the solution for the problem (1.1) and (1.2).

Lemma 2.3 (Auxiliary Lemma)**.** *Let* $\omega, z \in L[0, 1]$ *and* $x, y \in AC^2[0, 1]$. *Then the unique solution of the problem*

$$\begin{cases} ^cD^\alpha x(t) = \omega(t), \quad t \in [0, T], \quad 1 < \alpha \le 2, \\ ^cD^\beta y(t) = z(t), \quad t \in [0, T], \quad 1 < \beta \le 2, \\ x(0) = h(y), \quad \int_0^T y(s) ds = \mu_1 x(\eta), \\ y(0) = \phi(x), \quad \int_0^T x(s) ds = \mu_2 y(\xi), \end{cases} \tag{2.1}$$

is

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \omega(s) ds + (\sigma_1 t + 1) h(y) + \sigma_2 t \phi(x) \\ & + \frac{t}{\Delta} \Bigg[\mu_2 \xi \Bigg(\mu_1 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \omega(s) ds \\ & - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} z(s) ds \Bigg) \\ & + \frac{T^2}{2} \Bigg(\mu_2 \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} z(s) ds \\ & - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} \omega(s) ds \Bigg) \Bigg], \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} y(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} z(s) ds + (\sigma_3 t + 1) \phi(x) + t \sigma_4 h(y) \\ & + \frac{t}{\Delta} \Bigg[\mu_1 \eta \Bigg(\mu_2 \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} z(s) ds \\ & - \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} \omega(s) ds \Bigg) \\ & + \frac{T^2}{2} \Bigg(\mu_1 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \omega(s) ds \\ & - \int_0^T \frac{(T-s)^\beta}{\Gamma(\beta+1)} z(s) ds \Bigg) \Bigg], \end{aligned} \tag{2.3}$$

where

$$\Delta = \frac{T^4 - 4\mu_1\mu_2\eta\xi}{4}, \quad \sigma_1 = \frac{2\mu_1\mu_2\xi - T^3}{2\Delta},$$
$$\sigma_2 = \frac{T\mu_2(T-2\xi)}{2\Delta}, \quad \sigma_3 = \frac{2\mu_1\mu_2\eta - T^3}{2\Delta},$$