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Fixed points of fuzzy contractive and fuzzy locally contractive maps

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ABSTRACT

We establish some fixed point theorems for fuzzy contractive and fuzzy locally contractive mappings on a compact metric space with the d_∞-metric for fuzzy sets. Our results generalized well-known classical results of Edelstein.

1. Introduction and preliminaries

Since the appearance of celebrated Banach contraction principle in 1932, a variety of generalizations of this theorem in the setting of point to point mappings have been obtained. In 1969, Nadler [24] combined the idea of multivalued mappings and contractions. He proved some classical fixed points results for multivalued contractions. He also introduced the idea of multivalued locally contractions and generalized a fixed point theorem of Edelstein [11]. On the other hand Heilpern [18] introduced the notion of fuzzy mappings and established a fixed point theorem for fuzzy contraction mappings in a complete metric linear space, which is a fuzzy extension of Banach contraction principle and Nadler's [24] fixed point theorem. Afterwards, several authors [1–3,8,16,21,22,26,30] studied fixed point theorems for fuzzy generalized contractive mappings.

Edelstein [11,12] established a generalization of Banach contraction principle for mappings satisfying less restrictive conditions such as local contractions and locally contractive mappings. Subsequently, these mappings were studied by many others researchers [4–7,19,20,24,28,29,31] and a number of further extensions had been obtained.

Recently, Azam and Beg [3] obtained fuzzy fixed points of (ε, λ) uniformly fuzzy locally contractions and established fuzzy extension of Edelstein's fixed point theorem [11]. Aim of this paper is to study the existence of fixed points of fuzzy contractive and fuzzy locally contractive mappings, which are fuzzy generalizations of Edelstein's results in [12]. This approach (see also [1,9,13,21,26]) may have important applications in quantum particle physics, particularly in both string and ε^∞- theory, which were introduced and intensively studied by El Naschie who also applied it in high energy particle physics [13–15].

Throughout this paper, we shall use the following notations which have been recorded from [11,12,16,18,20,24,26,29,31].

Let (X,d) be a metric space. For $x,y \in X$, an ε-chain from x to y is a finite set of points $x_0, x_1, x_2, \ldots, x_n$ such that $x = x_0,\ x_n = y$ and $d(x_j, x_{j+1}) \leqslant \varepsilon$ for all $j = 0, 1, 2, \ldots, n-1$. Denote by $\mathfrak{C}(X)$ the totality of fuzzy sets $\mu : X \to [0,1] = I$ which satisfy that for each $\alpha \in I$ the α-cut of μ

$$[\mu]_\alpha = \{x : \mu(x) \geqslant \alpha\},$$

is nonempty compact in X.

Let $\mu_1, \mu_2 \in \mathfrak{C}(X)$. Then μ_1 is said to be included in μ_2, denoted by $\mu_1 \subseteq \mu_2$ if and only if $\mu_1(x) \leqslant \mu_2(x)$ for each $x \in X$. We denote (as in [2,3,8,16,18,22]) the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where χ_A is the characteristic function of the crisp set A.

Let $C(X) = \{A{:}A$ is nonempty compact subset of $X\}$. For $A, B \in C(X)$ and $\varepsilon > 0$ the sets $N^d(\varepsilon, A)$ and $E^d_{A,B}$ are defined as follows:

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$$N^d(\varepsilon, A) = \{x \in X : d(x, A) < \varepsilon\},$$
$$E^d_{A,B} = \{\varepsilon : A \subseteq N^d(\varepsilon, B), B \subseteq N^d(\varepsilon, A)\},$$

where $d(x, A) = \inf\{d(x, y) : y \in A\}$. Recall that the Hausdorff metric d_H on $C(X)$ induced by d is defined as

$$d_H(A, B) = \inf E^d_{A,B}.$$

For $\alpha \in I$, define

$$P_\alpha(A, B) = \inf_{x \in [A]_\alpha,\ y \in [B]_\alpha} d(x, y),$$
$$D_\alpha(A, B) = d_H([A]_\alpha, [B]_\alpha),$$

we write $P_\alpha(x, B)$ instead of $P_\alpha(\{x\}, B)$. If d^* is another metric on X then

$$P^*_\alpha(A, B) = \inf_{x \in [A]_\alpha,\ y \in [B]_\alpha} d^*(x, y),$$
$$D^*_\alpha(A, B) = d^*_H([A]_\alpha, [B]_\alpha).$$

Now define $d_\infty : \mathfrak{C}(X) \times \mathfrak{C}(X) \to \mathbb{R}$, (induced by the Hausdorff metric d_H) as

$$d_\infty(A, B) = \sup_\alpha D_\alpha(A, B).$$

We note that [26], d_∞ is a metric on $\mathfrak{C}(X)$ and the completeness of (X, d) implies that $(C(X), d_H)$ and $(\mathfrak{C}(X), d_\infty)$ are complete. Moreover, $(X, d) \mapsto (C(X), d_H) \mapsto (\mathfrak{C}(X), d_\infty)$ are isometrics embeddings by means of $x \to \{x\}$ (crisp set) and $A \to \chi_A$ respectively.

Let X be an arbitrary set, Y be a metric space. A mapping T is called fuzzy mapping if T is a mapping from X into set I^Y of all fuzzy sebsets of Y.

A mapping $T : X \to \mathfrak{C}(X)$ is called fuzzy (globally) contraction [18] if there exists $\lambda \in [0, 1)$ such that

$$d_\infty(T(x), T(y)) \leqslant \lambda d(x, y),$$

for all $x, y \in X$. Mapping T is said to be (ε, λ) uniformly fuzzy locally contraction [3] if

$$x, y \in X, d(x, y) < \varepsilon \ \Rightarrow d_\infty(T(x), T(y)) \leqslant \lambda d(x, y).$$

Mapping T is said to be fuzzy (globally) contractive (see [12,28]) if for all $x, y \in X,\ x \neq y$

$$d_\infty(T(x), T(y)) < d(x, y). \tag{1}$$

Mapping T is known as fuzzy locally contractive (see [12,28]) if each x of X belongs to an open set U so that if $y, z \in U,\ y \neq z$,

$$d_\infty(T(y), T(z)) < d(y, z). \tag{2}$$

A point $x \in X$ is said to be fuzzy fixed point of a fuzzy mapping T if

$$\{x\} \subset T(x).$$

Lemma 1.1 (Nadler [24]). *Let (X, d) be a metric space and $A, B \in C(X)$. Then for any $a \in A$ there exists a $b \in B$ such that*

$$d(a, b) \leqslant d_H(A, B).$$

2. Edelstein theorem for fuzzy contractive maps

One very useful and significant fixed point theorem, due to Edelstein [12] is that if (X, d) is a compact metric space and $T : X \to X$ is a contractive mapping (i.e. $d(Tx, Ty < d(x, y)$ for each $x, y \in X$). Then there exists a unique fixed point of T. Subsequently, Beg [6], Daffer and Kaneko [10], Grabiec [17], Hu and Rosen [20], Park [25], Rosenholtz [28], Mihet [23], Razani [27] Smithson [29], among others studied some extensions/generalizations and applications of this result.

Here by providing following theorem, we extend the above result [12] to fuzzy mappings.

Theorm 2.1. *Let (X, d) be a compact metric space and $T : X \to \mathfrak{C}(X)$ be a fuzzy (globally) contractive mapping. Then T has a fuzzy fixed point.*

Proof. For each $x \in X$, $[Tx]_1$ is nonempty and compact. Define a real valued function $g : X \to \mathbb{R}$ by

$$g(x) = P_1(x, T(x)).$$

It implies that