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Broad-scale small-world network topology induces optimal synchronization of flexible oscillators

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ABSTRACT

The discovery of small-world and scale-free properties of many man-made and natural complex networks has attracted increasing attention. Of particular interest is how the structural properties of a network facilitate and constrain its dynamical behavior. In this paper we study the synchronization of weakly coupled limit-cycle oscillators in dependence on the network topology as well as the dynamical features of individual oscillators. We show that flexible oscillators, characterized by near zero values of divergence, express maximal correlation in broad-scale small-world networks, whereas the non-flexible (rigid) oscillators are best correlated in more heterogeneous scale-free networks. We found that the synchronization behavior is governed by the interplay between the networks global efficiency and the mutual frequency adaptation. The latter differs for flexible and rigid oscillators. The results are discussed in terms of evolutionary advantages of broad-scale small-world networks in biological systems.

1. Introduction

Complex networks have attracted a great deal of interest since the discoveries of the small-world [1] and scale-free [2] properties. Signatures of such exceptional topological features have been discovered in many natural and man-made systems, which represents the root of modern network science [3,4]. It is nowadays a well-known fact that many real-world networks display degree distributions that deviate from a Poisson distribution found for simple random graph models. Generally three classes of small-world networks have been identified [5]. Very abundant types are scale-free networks (SFNs), whose degree distribution follows a power law. However, in many real networks, especially the physically embedded ones, there are different constraints which hinder the development of extremely connected nodes in a network, which leads to a cutoff of the power law regime in the connectivity distribution or making it disappear altogether. As a result, the so-called broad-scale networks (BSNs) and single-scale networks emerge [5].

With the progress in the network science it has been pointed out that networks can be treated not just as abstract entities with the vertices or nodes as formless place-holders, but as oscillators or dynamical systems coupled in the geometry of the network. Perhaps one of the most studied phenomena in this context is the synchronization of dynamical nodes in a network. Synchronization in networks is a very common collective behavior in real systems and is manifested by the appearance of some forms of relations between the functions of different dynamical variables as a result of interactions [6,7].

It is known that structural properties of a network significantly affect synchronization behavior. Primal works have found that complex networks, due to their small network distances, are generally more synchronizable than

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regular networks [7–10]. However, especially SFNs are characterized by heterogeneity in the degree distribution, which on one hand tends to reduce the average network distance, but on the other hand in more heterogeneous networks synchronization is increasingly harder to achieve [11]. This issue has been further addressed by several authors. Motter et al. [12] have revealed that directed and weighed coupling can balance the heterogeneity in the degree distribution. Furthermore, Zhou and Kurths [13] reported that in heterogeneous complex networks individual oscillators exhibit different levels of synchronization with respect to the collective dynamics and they exhibit a hierarchical dependence on the connection degrees. Moreover, it has been shown that the path to synchronized behavior is in homogeneous networks different than in heterogeneous network [14,15]. McGraw and Menzinger [16,17] have shown that the existence of community structures inside the network can hinder the synchronization of oscillators.

In our previous study [18] we observed that synchronization of oscillators in a network depends on its heterogeneity. We studied synchronization of Rössler oscillators on a spatially embedded network, where the level of heterogeneity could be varied from a highly heterogeneous SFN to a rather homogeneous random geometric network. It turned out that the maximal synchronization was obtained in the intermediate heterogeneous regime, i.e. in the BSN. The same results of this optimal intermediate network configuration were also obtained for the Rulkov map, where we additionally found that the flexibility of individual oscillators in the network might play an important role [19]. We further studied this phenomenon and found that the flexibility of individual oscillators in a network influences correlation and synchronization of oscillators also in other systems [20,21]; however, we did not succeed to explain this phenomenon until now.

In this paper we provide a mathematical explanation why flexible oscillators, characterized by low values of divergence, express maximal correlation in BSNs, whereas the non-flexible (rigid) oscillators are best correlated in more heterogeneous SFNs. To this purpose we use a mathematical model (Section 2) in which the dynamics of each node in the network is governed by the dynamics of the Poincaré oscillator. The network model enables smooth changes of the topology from a highly heterogeneous SFN to a homogeneous network. This setup enables us to explore the relationship between the flexibility of individual oscillators and the connectivity pattern that leads to the most synchronized response. The results (Section 3) show that correlations are maximized for rigid oscillators in a highly heterogeneous SFN, and for flexible oscillators in a less heterogeneous BSN. These findings are discussed in Section 4 in terms of a trade-off between high efficiency and an increased fragility of the networked systems, which could have an important role in the evolution of biological systems.

2. Mathematical model

We consider a network of coupled oscillators where the structure of this network can be varied from a highly heterogeneous SFN to a homogeneous regular network. For this purpose we utilize the spatially embedded vertex fitness model [19,22,23]. Initially each node is labeled with an integer i and a fitness value f_i is prescribed to it, where the fitness values follow a power-law distribution with a scaling exponent $\beta = 2.5$ [23]. Afterwards all N nodes are randomly distributed in a unit square. A connection between the i-th and j-th node is established if the following conditions is satisfied:

$$\Theta < \frac{f_i f_j}{l_{ij}^{\delta}}, \tag{1}$$

where Θ is used as the connectivity threshold that defines the average node degree of the network $\langle k \rangle$, l_{ij} is the Euclidean distance between the i-th and j-th node and the parameter δ alters topological features of the network. Following this algorithm we constructed the connectivity matrix with ij-th element d_{ij} equal to 1 if those two nodes are connected and 0 otherwise. If δ is near zero the connections are accepted only according to the fitness values independently of the inter-nodal Euclidean distances. In this case a highly heterogeneous SFN is constructed with mainly long-range connections. By increasing the topology parameter δ, long-range connections become more rare. The network topology becomes less heterogeneous, yet the presence of a few long-range connections still make the network very efficient in terms of small-world topological features [24]. Remarkably, the resulting constrains which limit the lengths of connections cause a cut-off in the power-law regime, meaning that extremely connected nodes are missing. Such networks are thus classified as BSNs [5]. On the other hand, for high values of the topology parameter $(\delta \gg 1)$ only short-range connections are accepted and the network becomes very homogeneous and inefficient.

In the network, the dynamics of each node is governed by the paradigmatic Poincaré oscillator:

$$\dot{x}_i = \gamma(A - r_i)x_i - 2\pi\nu_i y_i + \epsilon \sum_{j=1}^{N} d_{ij}(x_j - x_i), \tag{2}$$

$$\dot{y}_i = \gamma(A - r_i)y_i + 2\pi\nu_i x_i + \epsilon \sum_{j=1}^{N} d_{ij}(y_j - y_i), \tag{3}$$

where $\dot{x}_i$ and $\dot{y}_i$ are time derivatives of the variables x_i and y_i of the i-th oscillator, A is the limit cycle radius, $r_i = \sqrt{x_i^2 + y_i^2}$ is the distance from the origin to the attractor in the phase space, ν_i is the frequency, ϵ is the coupling strength and d_{ij} is the ij-th element of the connectivity matrix. The parameter γ represents the dissipation rate for the uncoupled Poincaré oscillator which indicates the stability of the oscillator with respect to amplitude perturbations [25,26] (see Appendix). We refer to oscillators with low values of the dissipation parameter γ as flexible and oscillators with high values of the parameter γ as rigid oscillators. Namely, the dissipation rate is directly related to the oscillators frequency adaptation to an external periodic signal [26–29], as flexible oscillators are easier to entrain.