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On the hitting depth in the dynamical system of continued fractions ☆

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ABSTRACT

Let $([0,1),T)$ be the dynamical system of continued fraction expansions with T the Gauss map. For any irrational point $y \in [0,1)$, let $I_n(y)$ be the cylinder of order n containing y. For any $x \in [0,1)$, define the maximal hitting depth of x to the irrational point y as

$$R_n(x,y) := \max\{t \in \mathbb{N}_0 : T^i(x) \in I_t(y) \text{ for some } 0 \leqslant i < n\}.$$

In this note, we investigate the asymptotic behaviors of $R_n(x,y)$ with respect to the Gauss measure as well as Hausdorff dimensions of exceptional sets related to the asymptotic properties of $R_n(x,y)$.

1. Introduction

Assume that $(X,T,\mathcal{B},\mu)$ is a measure theoretic dynamical system. Poincaré's recurrence theorem states that if A is a measurable set of positive measure, for μ-almost every point $x \in A$,

$$\sharp\{n \in \mathbb{N}_0 : T^n x \in A\} = \infty,$$

i.e. the forward orbit of x will hit the target A infinitely many times. Here $\mathbb{N} = \{1,2,3,\cdots\}$ denotes the set of natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbol $\sharp$ is used to denote the cardinality of a set.

Poincaré's recurrence theorem is a qualitative result in nature. A further quantitative study on the recurrence theorem leads to the study about the first return time.

Let A be a subset of X and $x \in X$. Define

$$\tau_A(x) = \min\{n \in \mathbb{N}_0 : T^n(x) \in A\},$$

i.e., the time needed for $x \in X$ to enter for the first time in A. In this aspect, one concerns about how the behavior of $\tau_A(x)$ depends on the mass of A.

The set A is always assumed two forms: (i) A is a ball $B(x,r)$ centered at x with radius r and one concerns the asymptotic behavior of $\tau_{B(x,r)}(x)$ when r decreases to zero; (ii) A is a ball $B(y,r)$ centered at another given point y and one concerns the asymptotic behavior of $\tau_{B(y,r)}(x)$.

Under the assumption that μ has a rapidly decaying correlation, it was shown by Saussol [19] that for μ-almost all x,

$$\liminf_{r\to 0} \log\frac{\log \tau_{B(x,r)}(x)}{-\log r} = \underline{d}_\mu(x), \quad \limsup_{r\to 0} \log\frac{\log \tau_{B(x,r)}(x)}{-\log r} = \overline{d}_\mu(x),$$

and by Galatolo [8] and by Galatolo & Kim [10] that for any $y \in X$ and μ-almost all x,

$$\liminf_{r\to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log r} = \underline{d}_\mu(y), \quad \limsup_{r\to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log r} = \overline{d}_\mu(y),$$

where $\underline{d}_\mu(\cdot)$ and $\overline{d}_\mu(\cdot)$ are the lower and upper pointwise dimension of the measure μ defined as

$$\underline{d}_\mu(\cdot) = \liminf_{r\to 0} \frac{\log \mu(B(\cdot,r))}{\log r},$$
$$\overline{d}_\mu(\cdot) = \limsup_{r\to 0} \frac{\log \mu(B(\cdot,r))}{\log r}.$$

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For related results, one is referred to [2,3,7,15,16,19,20] and the references therein.

The results given above provide a precise relation between the return time $\tau_{B(y,r)}(x)$ and the local dimension $d_\mu(y)$, and thus information can be inferred from one to the other. Roughly speaking, the information on the return time is much easier to obtain via observing the trajectories of the point x. Moreover, if we are in the setting of a symbolic space, the ball $B(y,r)$ in the real line is closely connected with the cylinder of almost same length containing y. And if we consider the cylinders containing y instead of the balls about y, it will be easier to understand when the orbit of a point x will fall into the cylinders containing y by just comparing the prefixes of the corresponding symbolic representations of the points x and y.

In this paper, we consider the dynamical system of continued fraction. This dynamics is uniformly expanding with infinitely many branches, and it also plays an important role in number theory due to its close connection with the classic Diophantine approximation.

Let us first fix some notations. Let T be the Gauss map on $[0,1)$ defined as

$$T(x) = 1/x - \lfloor 1/x \rfloor \text{ for } x \neq 0, \text{ and } T(0) = 0,$$

where $\lfloor\cdot\rfloor$ denotes the integer part of a real number. The Gauss measure μ is then defined to be $\mathrm{d}\mu(x) = \frac{1}{\log 2}\frac{1}{1+x}\mathrm{d}x$. It is well known that T preserves μ, and μ is ergodic [4].

For $x \in [0,1)$, the continued fraction expansion of

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots}}}$$

is always written to be a (finite or infinite) sequence $[a_1(x), a_2(x), \ldots]$, where $a_i(x) = \lfloor\frac{1}{T^i(x)}\rfloor \in \mathbb{N}$ for all $i \geqslant 1$, called the i-th partial quotient of x. Note that the continued fraction expansion of $x \in [0,1)$ is a finite sequence if and only if x is rational.

For an irrational point $y \in [0,1)$, define $I_0(y) = [0,1)$, and for any $n \geqslant 1$,

$$I_n(y) = \{x \in [0,1) : a_1(x) = a_1(y), \ldots, a_n(x) = a_n(y)\},$$

and call it the cylinder of order n containing the point y, where $a_1(y), \ldots, a_n(y)$ are the first n partial quotients of $y \in [0,1)$.

Given an irrational point $y \in [0,1)$, for any $x \in [0,1)$, define

$$R_n(x,y) := \max\left\{t \in \mathbb{N}_0 : T^i(x) \in I_t(y) \text{ for some } 0 \leqslant i < n\right\}.$$

We call $R_n(x,y)$ the maximal hitting depth of x to y, which reflects, to some extent, the degree how the trajectories of x can approach y.

It should be mentioned that the continued fraction dynamical system $([0,1), T)$ with respect to the Gauss measure has ψ-mixing properties (see Proposition 2.3 below). Hence the results of Saussol [19] and Galatolo [8] are all valid for this special system.

Moreover, the quantity $R_n(x,y)$ is very closely related to the one

$$d_n(x,y) = \min_{1\leqslant i\leqslant n} d(T^i(x), y),$$

in a metric space (X,d), introduced by Galatolo & Peterlongo [9] to study $\tau_{B(y,r)}(x)$. From the definitions of $R_n(x,y)$ and $d_n(x,y)$, one can see that on one hand, if $R_n(Tx,y) = k$, then, in light of Proposition 2.2 below, we have

$$|I_{k+3}(y)| \leqslant d_n(x,y) \leqslant |I_k(y)|;$$

on the other hand, letting $k \in \mathbb{N}$ be the integer such that $|I_{k+1}(y)| \leqslant d_n(x,y) < |I_k(y)|$, we have

$$k-2 \leqslant R_n(Tx,y) \leqslant k+1.$$

As a result, if the Lyapunov exponent $\lambda(y) := \lim_{n\to\infty}\frac{\log|(T^n)'(y)|}{n}$ of T at y exists, which is equal to $\lim_{n\to\infty}\frac{-\log|I_n(y)|}{n}$ in the continued fraction case, then together with the relationship between $d_n(x,y)$ and $\tau_{B(y,r)}(x)$ ([9]), we have, for all x, that

$$\lambda(y)\cdot\liminf_{n\to\infty}\frac{R_n(x,y)}{\log n} = \liminf_{n\to\infty}\frac{-\log d_n(x,y)}{\log n} = \limsup_{r\to 0}\frac{-\log r}{\log\tau_{B(y,r)}(x)}$$

and

$$\lambda(y)\cdot\limsup_{n\to\infty}\frac{R_n(x,y)}{\log n} = \limsup_{n\to\infty}\frac{-\log d_n(x,y)}{\log n} = \liminf_{r\to 0}\frac{-\log r}{\log\tau_{B(y,r)}(x)}.$$

Therefore, if $\lambda(y)$ exists, then for almost all $x \in [0,1)$, the asymptotic behavior of $R_n(x,y)$ can be studied by the results from [8–10].

Instead of the above almost-sure type results, in this paper, we intend to get general results by studying the behavior of $R_n(x,y)$ at any irrational point y. We will see that, compared with the asymptotic behavior of $\tau_{B(y,r)}(x)$, the quantity $R_n(x,y)$ is related to the so-called local entropy. The lower local entropy of y with respect to the Gauss measure μ is defined as

$$\underline{D}_\mu(y) = \liminf_{n\to\infty}\frac{-\log\mu(I_n(y))}{n},$$

and the upper one is defined similarly by replacing lim inf by lim sup.

We prove that.

Theorem 1.1. *For any irrational point $y \in [0,1)$, for almost all $x \in [0,1)$,*

$$\limsup_{n\to\infty}\frac{R_n(x,y)}{\log n} = \frac{1}{\underline{D}_\mu(y)}, \quad \liminf_{n\to\infty}\frac{R_n(x,y)}{\log n} = \frac{1}{\overline{D}_\mu(y)}.$$

We remark that by Shannon–McMillan–Breiman theorem $\underline{D}_\mu(y) = \overline{D}_\mu(y)$ for μ-a.e. $y \in [0,1)$, and thus the limit $\lim_{n\to\infty}\frac{R_n(x,y)}{\log n}$ exists for such y.

This result provides a way to calculate the local entropy: in order to determine the local entropy of y, one just needs to know the value of $R_n(x,y)$ for the typical point x, which can be obtained by comparing the corresponding continued fraction representations of x and y.