Chaos, Solitons & Fractals 69 (2014) 117–128

Contents lists available at ScienceDirect

Chaos, Solitons & Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Self-similar sets satisfying the common point property

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ARTICLE INFO

Article history:
Received 26 May 2014
Accepted 12 September 2014
Available online 9 October 2014

ABSTRACT

A self-similar set is a fixed point of iterated function system (IFS) whose maps are similarities. We say that a self-similar set satisfies the common point property if the intersection of images of the attractor under the maps of the IFS is a singleton and this point has a common pre-image, under the maps of the IFS, and the pre-image is in the attractor.

Self-similar sets satisfying the common point property were introduced in Sirvent (2008) in the context of space-filling curves. In the present article we study some basic topological and dynamical properties of self-similar sets satisfying the common point property. We show examples of this family of sets.

We consider attractors of a sub-IFS, an IFS formed from the original IFS by removing some maps. We put conditions on this attractors for having the common point property, when the original IFS have this property.

1. Introduction

A self-similar set is a set that can be decomposed into subsets that are similar copies of the original set. A self-similar set can be expressed as the fixed point of an iterated function system, or IFS (for short), whose maps are similarities. This framework was introduced by Hutchinson [13]. Many fractals are self-similar sets and, more generally, fixed points of IFSs, whose maps are contractions but not necessary similitudes. Self-similar sets have been extensively studied, see for instance [16] or [14] and references within.

The topology and geometry of a self-similar set depends of the structure of the overlaps of the IFS, among other references see [5,6,12]. In the present article we study self-similar sets that satisfy the common point property. This is a particular type of overlap. We say that a fixed point of an IFS satisfies the common point property, if the intersection of images of the attractor under the maps of the IFS is a singleton and this point has a common pre-image, under the maps of the IFS, and the pre-image is in the attractor. These sets were introduced in [24], in the context of studying dynamical and geometrical properties of space filling curves. The importance of these sets is that we can associate a space filling curve (of the set) and a geodesic lamination that is compatible with the dynamics, defined by the IFS of the self-similar set. The classical space-filling curves, i.e. Peano [18,19], Hilbert [11], etc., do not fit into this category. A space-filling curve that fit into this class, was presented in [1], for further studies of this curve see [2,23].

In Section 2, we introduce the precise definitions and show some basic properties of the sets that satisfy the common point property. In Theorem 2.1 we give some conditions for the uniqueness of the address of the point corresponding to the pre-image of the common point. Which is an important fact in order to characterize these sets. We show that the common point property is invariant under involutions that fix the set and the pre-image of the common point (Proposition 2.2).

In Section 3, we give an extensive list of examples.

In Section 4, we introduced the concept of a sub-IFS, of a given IFS, and its corresponding attractor. This object was used in [25], in the study of symmetries of geodesic laminations associated to space-filling curves. In this

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http://dx.doi.org/10.1016/j.chaos.2014.09.011
0960-0779/

section we discuss when the attractor of a sub-IFS, of an IFS satisfying the common point property, inherits this property. In Theorem 4.1 we give some conditions for that.

Section 5 is dedicated to open problems and remarks. Here we conjecture that all the self-similar sets in the plane satisfying the common point property, having positive Lebesgue measure and satisfying an extra technical condition, are the ones listed in the article.

We end the article with Appendix A, where we summarize the construction of space-filling curves, associated with self-similar sets satisfying the common point property. The results of this section where developed in [24,25].

2. Basic definitions and properties

An *iterated function system or IFS* consists of a complete metric space X together with a finite set of contraction mappings, $\{\phi_1,\ldots,\phi_k\}$. Let $K(X)$ be the set of all non-empty compact subsets of X, we give to this set the Hausdorff metric. With this distance $K(X)$ is a complete metric space [7,10]. We define the map $\Phi : K(X) \to K(X)$, as $\Phi(U) = \cup_{i=1}^{k}\phi_i(U)$. This map is called the *Hutchinson operator* (*cf.* [13]). Since the maps ϕ_i are contractions the map Φ is a contraction on $K(X)$. Its fixed point is called the attractor of the IFS $\{\phi_1,\ldots,\phi_k\}$. Let F be the attractor of an IFS, so by its definition, it is the unique non-empty compact set that satisfies the set equation:

$$F = \phi_1(F) \cup \cdots \cup \phi_k(F). \tag{1}$$

Let $\{\phi_1,\ldots,\phi_k\}$ be an IFS on $\mathbb{R}^d$ with $d \geqslant 2$, such that $\phi_i(x) = A_i x + \mathbf{v}_i$, where $A_i = \rho_i O_i$ is a matrix with $0 < \rho_i < 1$ and O_i a $d \times d$–orthogonal matrix, and $\mathbf{v}_i \in \mathbb{R}^d$. The attractor of the IFS is called a *self–similar set*.

Let F be the attractor of an IFS, we say that the set F (or the IFS) satisfies the *common point property or CPP* if $\cap_{i=1}^{k}\phi_i(F)$ consists of only one point, say z, and there exists one point $y \in F$ such that $\phi_1(y) = \phi_2(y) = \cdots = \phi_k(y) = z$. The point z is called the *intersection point of F* and the point y the *pre-image of the intersection point by the IFS*.

Some basic topological properties of these sets can be proved easily. By [12] the attractors that satisfy the common point property are connected. On the other hand, we have the following proposition:

Proposition 2.1. *Let F be a self-similar set satisfying the CPP. The pre-image of the intersection point of F, i.e. the point y, belongs to the topological boundary of F.*

Proof. Since the intersection of the sets $\phi_j(F)$ consists of only one point, i.e. z, and the maps ϕ_j are continuous, we have

$$\{z\} = \bigcap_{j=1}^{k}\phi_j(\partial F),$$

where ∂F is the topological boundary of F. By definition $y = \phi_j^{-1}(z)$, for $1 \leqslant j \leqslant k$, hence $y \in \partial F$. □

We say that the IFS satisfies the *open set condition or OSC* if there exists a non-empty bounded open set V such that $\cup_{i=1}^{k}\phi_i(V) \subset V$ with disjoint union (*cf.* [10]).

Throughout the article we use the notation $F_{a_1\ldots a_n}$ for $\phi_{a_1} \circ \cdots \circ \phi_{a_n}(F)$.

Each point of F can be obtained as $\cap_1^\infty F_{a_1\ldots a_n}$ for a suitable infinite sequence $a_1a_2\ldots$ in $\{1,\ldots,k\}^{\mathbb{N}}$. This sequence is called an *address* of the point. The address might not be unique. From the definition follows that the sequence $a_1a_2\cdots$ is the only address of the point $x \in F$ if and only if

$$\forall n \geqslant 1,\ x \in F_{a_1\ldots a_{n-1}a_n} \text{ and } x \notin F_{a_1\ldots a_{n-1}j} \text{ for } j \neq a_n. \tag{2}$$

We formalize these ideas as follows: Let $\Sigma = \{1,\ldots,k\}^{\mathbb{N}}$ and $\pi : \Sigma \to F$, given by

$$\{\pi(a_1a_2\ldots)\} := \bigcap_{1}^{\infty}(\phi_{a_1} \circ \cdots \circ \phi_{a_n}(F)). \tag{3}$$

Since the set F is the attractor of the IFS $\{\phi_1,\ldots,\phi_k\}$, the map π is surjective. If we endowed Σ with the product topology, the map π is continuous, since the maps ϕ_j are continuous. The map π is called the *projection map* of the IFS. An address of a point $x \in F$ is an element of $\pi^{-1}(x)$.

Let $\mathcal{O} \subset F$ defined by $\mathcal{O} = \cup_{i\neq j}F_i \cap F_j$. The set $\mathcal{O}$ is called the *overlap set* of the IFS. Clearly the address of every point of F is unique if and only if the overlap set is empty. Let $\mathcal{N}$ be the set of points whose addresses are not unique. It corresponds to the points of $\mathcal{O}$ and its orbit under the IFS, i.e.

$$\mathcal{N} := \left(\bigcup_{n\geqslant 1}\{\phi_{a_1} \circ \cdots \circ \phi_{a_n}(\mathcal{O}) : 1 \leqslant a_i \leqslant k, 1 \leqslant i \leqslant n\}\right)\bigcup \mathcal{O}.$$

Let $\sigma : \Sigma \to \Sigma$ be the *shift map*, defined by $\sigma(a_1a_2\ldots) = a_2\ldots$. If the overlap set is empty we can define the map $T : F \to F$ by $T(x) = \pi \circ \sigma \circ \pi^{-1}(x)$, this map is called the *geometric shift* (*cf.* [17]). If $\mathcal{O} \neq \emptyset$, the map T is not defined as a point map, but it defines a set valued map. By the definition of T, we have that if $x \in F$ with $T^{n-1}(x) \cap F_{a_n} \neq \emptyset$, for all $n \geqslant 1$ then $\pi(a_1a_2\ldots) = x$.

From the definition of F and $\mathcal{N}$ follows that the set $\mathcal{N} = \cup_{n\geqslant 0}T^{-n}(\mathcal{O})$.

The concept of *dynamical boundary* of an attractor of an IFS, was introduced by M. Morán in [17]. We denote the dynamical boundary of F by $\partial_D(F)$. It is defined as the closure of all the images of the overlap set under the geometric shift map, i.e.

$$\partial_D(F) = \overline{\bigcup_{n\geqslant 1}T^n(\mathcal{O})}.$$

If $F \subset \mathbb{R}^d$ has non-empty interior and satisfies the open set condition, then the dynamical boundary of F is its topological boundary [17].

If the set F satisfies the CPP, then y is in the dynamical boundary of F; since $z \in \mathcal{O}$ and $y \in T(\mathcal{O})$.

We say that a point $x \in F$ is *periodic (or T-periodic)* if there exists a non-negative integer m such that $\{x\} \subseteq T^m(x)$. Equivalently there exists $\mathbf{a} = a_1a_2\cdots \in \Sigma$, such that $\sigma^m(\mathbf{a}) = \mathbf{a}$ and $\pi(\mathbf{a}) = x$. For a point $\mathbf{a} \in \Sigma$ we say it is *σ-periodic* if $\sigma^m(\mathbf{a}) = \mathbf{a}$, for some non-negative m. We will denote the σ-periodic point

$$\mathbf{a} = a_1\cdots a_m a_1\cdots a_m a_1\cdots a_m\cdots$$

by $\overline{a_1\cdots a_m}$. And we say that $x \in F$ is *pre-periodic (or T–pre-periodic)*, if it is not periodic, and there exists a positive