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The properties of a class of biorthogonal vector-valued nonseparable bivariate wavelet packets ☆

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ABSTRACT

In this paper, we introduce a class of vector-valued wavelet packets of space $L^2(R^2, C^\kappa)$, which are generalizations of multivariate wavelet packets. A procedure for constructing a class of biorthogonal vector-valued wavelet packets in higher dimensions is presented and their biorthogonality properties are characterized by virtue of matrix theory, time–frequency analysis method, and operator theory. Three biorthogonality formulas regarding these wavelet packets are derived. Moreover, it is shown how to gain new Riesz bases of space $L^2(R^2, C^\kappa)$ from these wavelet packets. Relation to some physical theories such as the Higgs field is also discussed.

1. Introduction

It is well-known that matter is composed of few fundamental building blocks, known as quarks and leptons or generally fermions. A concise theoretical framework for the forces between these elementary particles is based on the principle of gauge invariance [1–5]. This principle made it possible to unify the electromagnetic and weak theory into a single electroweak interaction, and to develop a quantum field theory for the strong interaction, called quantum chromodynamics. These forces are mediated by the photon, the W and Z bosons and the gluons, the particles or quanta of the corresponding bosonic fields [3]. However, such theory of matter particles and force fields suffers from a serious deficiency, that is, the underlying gauge principle requires at first sight all quanta fields to be massless. At present, the only compelling way to achieve that is to invoke the so-called Higgs mechanism that gives the particles their masses while preserving gauge invariance. The basic idea is that a priori massless particles acquire effective masses by interacting with the Higgs boson associated with the Higgs field [1,2]. The basis of the so-called standard model of particle physics are the matter particles, the force fields of the electroweak theory and the Higgs mechanism, which is an extremely successful theory that has been tested and validated with high precision in a broad range of experiments. Only the Higgs particle has so far escaped experimental observation [1]. Therefore, one of the most pressing challenges of particle physics is to establish the Higgs mechanism, i.e., to find the Higgs particle experimentally and to study its properties, or to find an alternative explanation of the masses of the particles. In all events, it is very important to estimate the expected mass of the Higgs.

General arguments clearly point to the existence of an even more fundamental theory; supersymmetry is the favored idea underlying such an extension of the standard model. It leads to a consistent and calculable theory in which the Higgs mechanism can be accommodated in a natural way [4,5]. In supersymmetry [6], for every particle, there is a super-partner whose spin differs by 1/2. Most importantly, supersymmetry provides a framework for the unification of the electromagnetic, weak and strong forces at large energies. It is deeply related to gravity, the fourth in the fundamental forces. Supersymmetry

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predicts several Higgs particles with suggested mass below 200 GeV. The discovery of supersymmetry and the precision measurements of the properties of supersymmetric particles could provide a glimpse of the fundamental theory in which gravity is unified with the weak, electromagnetic and strong forces [4].

The last two decades or so have witnessed the development of wavelet theory. The main advantage of wavelets is their time–frequency localization property. Already they have led to exciting applications in signal analysis [7,8], fractals [9,10] and image processing [11], and so on. Many signals in areas like music, speech and images can be efficiently represented by wavelets. Wavelet packets, due to their nice characteristics, have been widely applied to signal processing [12], code theory [13], image compression [14], solving integral equation and so on. Coifman and Meyer first introduced the notion of univariate orthogonal wavelet packets. Cohen and Daubechies introduced biorthogonal wavelet packets [15]. Yang [16] constructed a-scale orthogonal multiwavelet packets which were more flexible in applications. Vector-valued wavelets are a class of special multiwavelets. Chen [17] introduced the notion of orthogonal vector-valued wavelets and discussed the properties orthogonal vector-valued wavelet packets [18], However, vector-valued wavelets and multiwavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms [19] but not necessary for discrete vector-valued transforms. Examples of vector-valued signals are video images. Therefore, it is useful for us to study vector-valued wavelets in representations of signals. It is known that the majority of information is multidimensional information. Shen [20] introduced multivariate orthogonal wavelets which may be used in a wider field. Thus, it is necessary to generalize the concept of multivariate wavelet packets to the case of multivariate vector-valued wavelets. The goal of this paper is to give the definition and the construction of biorthogonal vector-valued wavelet packets and construct several new Riesz bases of space $L^2(R^2, C^\kappa)$.

This paper is organized as follows. In Section 2, several notations and definitions concerning vector-valued function space are presented. In Section 3, we introduce vector-valued multiresolution analysis and vector-valued wavelets. In 4, we characterize the biorthogonality property of the vector-valued wavelet packets. In the final section, the wavelet packet bases of space $L^2(R^2, C^\kappa)$ are presented.

2. Notations and fundamentals on vector-valued function space

We start from introducing several notations and definitions. R and C stand for all real and all complex numbers, respectively. Z and N denote the set of integers and positive integers, respectively. Set $Z_+ = \{0\} \cup N$, and $a, \kappa \in N$ as well as $a, \kappa \geqslant 2$. $Z^2 = \{(z_1, z_2) : z_1, z_2 \in Z\}$, $Z_+^2 = \{(\eta_1, \eta_2) : \eta_1, \eta_2 \in Z_+\}$. For any $Y, Y_1, Y_2 \subset R^2$, let $aY = \{ay : y \in Y\}$, $Y_1 + Y_2 = \{y_1 + y_2 : y_1 \in Y_1, y_2 \in Y_2\}$, $Y_1 - Y_2 = \{y_1 - y_2 : y_1 \in Y_1, y_2 \in Y_2\}$. There exist a^2 elements $\mu_0, \mu_1, \ldots, \mu_{a^2-1}$ in Z_+^2 by the finite group theory such that $Z^2 = \cup_{\mu\in\Gamma_0}(\mu + aZ^2), (\mu_1 + aZ^2) \cap (\mu_2 + aZ^2) = \emptyset$, where $\Gamma_0 = \{\mu_0, \mu_1, \ldots, \mu_{a^2-1}\}$ denotes the set of all different representative elements in the quotient group Z^2/aZ^2 and μ_1, μ_2 denote two arbitrary different elements in Γ_0. Set $\mu_0 = \underline{0}$, where $\underline{0}$ is the zero element of Z_+^2. Let $\Gamma = \Gamma_0 - \{\underline{0}\}$ and Γ, Γ_0 be two index sets. By $L^2(R^2, C^\kappa)$, we denote the aggregate of all vector-valued functions $F(y)$, i.e., $L^2(R^2, C^\kappa) := \{F(y) = (f_1(y), f_2(y), \ldots, f_\kappa(y))^{\mathrm{T}} : f_\iota(y) \in L^2(R^2), \iota = 1, 2, \ldots, \kappa\}$, where T means the transpose of a vector. Video images are examples of vector-valued functions where $f_\iota(y)$ in the above $F(y)$ denotes the pixel on the ιth column at the point y. For $F(y) \in L^2(R^2, C^\kappa)$, $\|F\|$ denotes the norm of vector-valued function $F(y)$, i.e., $\|F\| := \left(\sum_{\iota=1}^{\kappa} \int_{R^2} |f_\iota(y)|^2 \,\mathrm{d}y\right)^{1/2}$, and its integration is defined to be

$$\int_{R^2} F(y)\mathrm{d}y := \left(\int_{R^2} f_1(y)\mathrm{d}y, \int_{R^2} f_2(y)\mathrm{d}y, \ldots, \int_{R^2} f_\kappa(y)\mathrm{d}y\right)^{\mathrm{T}}. \tag{1}$$

The Fourier transform of $F(y)$ is defined as $\widehat{F}(\gamma) := \int_{R^2} F(y)\mathrm{e}^{-\mathrm{i}y\cdot\gamma}\,\mathrm{d}y = \left(\int_{R^2} f_1(y)\mathrm{e}^{-\mathrm{i}y\cdot\gamma}\,\mathrm{d}y, \int_{R^2} f_2(y)\mathrm{e}^{-\mathrm{i}y\cdot\gamma}\,\mathrm{d}y, \ldots, \int_{R^2} f_\kappa(y)\mathrm{e}^{-\mathrm{i}y\cdot\gamma}\,\mathrm{d}y\right)^{\mathrm{T}}$, where $y \cdot \gamma$ stands for the inner product of real vectors y and γ. For $F, G \in L^2(R^2, C^\kappa)$, their *symbol inner product* is defined by

$$\langle F(\cdot), G(\cdot)\rangle := \int_{R^2} F(y)G(y)^{*}\mathrm{d}y, \tag{2}$$

where the superscript * means the transpose and the complex conjugate.

Definition 1. We say that two vector-valued functions $F(y)$, $\widetilde{F}(y) \in L^2(R^2, C^\kappa)$ are a pair of biorthogonal ones, if their translates satisfy

$$\langle F(\cdot), \widetilde{F}(\cdot - v)\rangle = \delta_{\underline{0},v} I_\kappa, \qquad v \in Z^2, \tag{3}$$

where I_κ denotes the $\kappa \times \kappa$ identity matrix and $\delta_{\underline{0},v}$ is the generalized Kronecker symbol, i.e., $\delta_{\underline{0},v} = 1$ when $v = \underline{0}$ and $\delta_{\underline{0},v} = 0$, otherwise.

Definition 2. A sequence of vector-valued functions $\{F_v(y)\}_{v\in Z^2} \subset U \subset L^2(R^2, C^\kappa)$ is called a Riesz basis of subspace V, if it satisfies the following two conditions: (i) for any $G(y) \in U$, there exists a unique $\kappa \times \kappa$ matrix sequence $\{B_v\}_{v\in Z^2} \in \ell^2(Z^2)^{\kappa\times\kappa}$ such that

$$G(y) = \sum_{v\in Z^2} B_v F_v(y), \qquad y \in R^2, \tag{4}$$