Chaos, Solitons and Fractals 41 (2009) 2234–2240

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

The asymptotic average-shadowing property and transitivity for flows ☆

Rongbao Gu

School of Finance, Nanjing University of Finance and Economics, Nanjing 210046, PR China

ARTICLE INFO

Article history:
Accepted 22 August 2008

Communicated by Prof. Ji-Huan He

ABSTRACT

The asymptotic average-shadowing property is introduced for flows and the relationships between this property and transitivity for flows are investigated. It is shown that a flow on a compact metric space is chain transitive if it has positively (or negatively) asymptotic average-shadowing property and a positively (resp. negatively) Lyapunov stable flow is positively (resp. negatively) topologically transitive provided it has positively (resp. negatively) asymptotic average-shadowing property. Furthermore, two conditions for which a flow is a minimal flow are obtained.

1. Introduction

Suppose (X,d) is a compact metric space with metric d. Write $\mathbb{R}=(-\infty,+\infty)$. Let $\varphi:\ \mathbb{R}\times X\to X$ be a flow, that is, φ is a continuous map and satisfies the following conditions:

(1) $\varphi(0,x)=x$ for all $x\in X$,
(2) $\varphi(s,\varphi(t,x))=\varphi(t+s,x)$ for all $x\in X$ and all $s,t\in\mathbb{R}$.

For $x\in X$, the set $\gamma(x,\varphi)=\{\varphi(t,x):t\in\mathbb{R}\}$ is said to be the orbit of φ through the point x. For $E\subseteq X$ and $t\in\mathbb{R}$, we write $\varphi(t,E)=\{\varphi(t,x):x\in E\}$.

A subset A in X is said to be invariant under φ if $\varphi(t,A)\subseteq A$ for all $t\in\mathbb{R}$, and A is said to be a minimal set of φ if it is non-empty, closed and invariant under φ and it does not contain any proper subset having these three properties.

A flow φ is said to be a minimal flow if X is a unique minimal set of φ. It is easy to see from compactness of X that φ is a minimal flow if and only if, for every point x in X, the orbit $\gamma(x,\varphi)$ of φ through x is dense in X.

As pointed out by Smale [1], a very important problem in dynamical systems theory is to find the minimal set. A brief summarization on this subject was given in [2]. The notion of the shadowing property arises from the study related to Anosov diffeomorphisms [3]. There are lots of existing works on finding the minimal set in the systems with the shadowing property. For example, Kato [4] showed that a Lyapunov stable flow with the shadowing property is a minimal flow; Komouro [5] showed almost at the same time that an equidistant flow with the shadowing property is a minimal flow; He and Wang [2] showed that a distal flow with the shadowing property is a minimal flow; most recently, Mai [6] showed that a pointwise recurrent flow with the shadowing property is a minimal flow.

As a certain generalization of the shadowing property in random dynamical systems, Blank [7] introduced the notion of the average-shadowing property in studying chaotic dynamical systems, which is a good tool to characterize Anosov diffeomorphisms (see [8]). In a recent work, the author [9] showed that a Lyapunov stable flow with the average-shadowing property is a minimal flow.

☆ Project supported by the Natural Science Foundation (05KJB110033) and the Philosophy and Social Science Foundation (07SJB790013) of Education Department of Jiangsu Province of China and the Science Foundation of Nanjing University of Finance and Economics (2007ZCB004).

E-mail address: rbgu@sina.com

0960-0779/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2008.08.029

Eirola et al. [10] posed the notion of the limit-shadowing property. From the numerical point of view this property of a dynamical system means the following: if we apply a numerical method that approximates the system with "improving accuracy" so that one step errors tend to zero as time goes to infinity, then the numerically obtained orbits tend to real ones. Researches in the limit-shadowing property see for instance [11].

In the present paper, we introduce the notion of the asymptotic average-shadowing property (Abbrev. AASP) for flows, which is a certain generalization of the limit-shadowing property in random dynamical systems. In Section 2, we introduce some basic terminology. In Section 3, we study the relationship between the AASP and chain transitivity. The relationship between the AASP and topological transitivity will be investigated in Section 4.

2. Some basic terminology

Given $\delta > 0$ and $T > 0$, a bi-sequence $(\{x_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ is said to be (δ, T)-pseudo-orbit of φ if $t_i \geqslant T$ and $d(\varphi(t_i, x_i), x_{i+1}) \leqslant \delta$ for all $i \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers.

A bi-sequence $(\{x_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ is said to be ε-shadowed by the orbit of φ through x, if there is an orientation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\varphi(\alpha(t), x), \varphi(t - s_i, x_i)) < \varepsilon, \quad \text{for } s_i \leqslant t < s_{i+1}, \quad i \geqslant 0$$

and

$$d(\varphi(\alpha(t), x), \varphi(t + s_{-i}, x_{-i})) < \varepsilon, \quad \text{for } -s_{-i} \leqslant t < -s_{-i+1}, \quad i \geqslant 1$$

where $s_0 = 0, s_n = \sum_{i=0}^{n-1} t_i, s_{-n} = \sum_{i=-n}^{-1} t_i, n = 1, 2, \ldots$.

A flow φ is said to have the shadowing property (or pseudo-orbit tracing property) if for any $\varepsilon > 0$ there is a $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit of φ can be ε-shadowed by some orbit of φ.

A flow φ is said to have the positive (resp. negative) limit-shadowing property if for every bi-sequence $(\{x_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ with $t_i \geqslant 1$ for all $i \in \mathbb{Z}$ and

$$\lim_{|i|\to\infty} d(\varphi(t_i, x_i), x_{i+1}) = 0,$$

there is an orientation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$\lim_{i\to\infty} \int_{s_i}^{s_{i+1}} d(\varphi(\alpha(t), x), \varphi(t - s_i, x_i))\mathrm{d}t = 0$$

$$\left(\text{resp.} \quad \lim_{i\to\infty} \int_{-s_{-i}}^{-s_{-(i-1)}} d(\varphi(\alpha(t), x), \varphi(t + s_{-i}, x_{-i}))\mathrm{d}t = 0\right).$$

We say that a flow φ has the limit-shadowing property if φ has both positive and negative limit-shadowing property.

In general, the shadowing property does not imply the limit-shadowing property and the limit-shadowing property also does not imply the shadowing property, see for instance [11,12].

As a generalization of the limit-shadowing property in random dynamical systems, the author posed in [13] the notion of the asymptotic average-shadowing property for discrete dynamical systems. In the following, we introduce this property for flows.

Definition 2.1. A bi-sequence $(\{x_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ is said to be an asymptotic T-average-pseudo-orbit of φ if $t_i \geqslant T$ for all $i \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{n} d(\varphi(t_i, x_i), x_{i+1}) = 0.$$

For convenience, we say asymptotic average-pseudo-orbit as asymptotic T-average-pseudo-orbit with $T = 1$.

A bi-sequence $(\{x_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ is said to be positively (resp. negatively) asymptotically shadowed in average by the orbit of φ through x, if there is an orientation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} d(\varphi(\alpha(t), x), \varphi(t - s_i, x_i))\mathrm{d}t = 0$$

$$\left(\text{resp.} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \int_{-s_{-i}}^{-s_{-(i-1)}} d(\varphi(\alpha(t), x), \varphi(t + s_{-i}, x_{-i}))\mathrm{d}t = 0\right),$$

where $s_0 = 0, s_n = \sum_{i=0}^{n-1} t_i, s_{-n} = \sum_{i=-n}^{-1} t_i, n = 1, 2, \ldots$

A flow φ is said to have the positively (resp. negatively) asymptotic average-shadowing property (Abbrev. PAASP (resp. NAASP)) if for every asymptotic average-pseudo-orbit of φ can be positively (resp. negatively) asymptotically shadowed in average by some orbit of φ.