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Limit cycle bifurcations in a class of near-Hamiltonian systems with multiple parameters ☆

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ABSTRACT

This article investigates a class of near-Hamiltonian systems and obtains some new conditions for the existence of multiple limit cycles with the help of the first order Melnikov function. As applications to the obtained main results, a cubic reversible isochronous system under cubic polynomial perturbations is studied.

1. Introduction and main results

Consider a near-Hamiltonian system of the form

$$\begin{aligned}\dot{x} &= H_y(x,y,\lambda) + \epsilon p(x,y,\epsilon,\lambda),\\ \dot{y} &= -H_x(x,y,\lambda) + \epsilon q(x,y,\epsilon,\lambda),\end{aligned} \tag{1.1}$$

where $0 < \epsilon \ll \lambda \ll 1$ and $H(x,y,\lambda),\ p(x,y,\epsilon,\lambda),\ q(x,y,\epsilon,\lambda)$ are C^∞ functions. Suppose the unperturbed vector field has a family of periodic orbits for λ small given by

$$L_\lambda(h):\ H(x,y,\lambda) = h, \quad h \in J_\lambda, \tag{1.2}$$

which reduces to

$$L_0(h):\ H(x,y,0) = h, \quad h \in J_0 = (\alpha, \beta)$$

for $\lambda = 0$. Eq. (1.1) has a displacement function of the form for ϵ small

$$d(h,\epsilon,\lambda) = \epsilon M(h,\lambda) + O(\epsilon^2), \tag{1.3}$$

where

$$M(h,\lambda) = \oint_{L_\lambda(h)} dx - pdy\Big|_{\epsilon=0}. \tag{1.4}$$

It has a representation as a power series in λ

$$M(h,\lambda) = M_0(h) + \lambda M_1(h) + \ldots + \lambda^n M_n(h) + O(\lambda^{n+1}), \tag{1.5}$$

which is convergent for small λ, where

$$M_0(h) = M(h,0) = \oint_{L_0(h)} qdx - pdy\Big|_{\epsilon=\lambda=0}. \tag{1.6}$$

Apparently, by the implicit function theorem, the zeros of (1.4) give some information about the zeros of (1.3) (see [4–12]), which determine the number of limit cycles of (1.1) emerging from $L_\lambda(h)$. The function $M(h,\lambda)$ in (1.4) is called the first order Melnikov function. The weakened Hilbert 16th problem, originally posed by Arnold [1] in 1977, is to find an upper bound of the number of zeros of (1.4) for polynomials $p,\ q$ and H.

If M in (1.5) has the form

$$M(h,\lambda) = \lambda^n \tilde{M}(h,\delta) + O(\lambda^{n+1}), \tag{1.7}$$

where $\delta \in D \subset R^m$ with D compact, then the number of limit cycles of Eq. (1.1) can be obtained by studying the number of zeros of the function $\tilde{M}(h,\delta)$ for $h \in (\alpha,\beta)$.

In particular, if we further suppose that $L_0(h)$ approaches an elementary center point of Eq. (1.1). $|_{\epsilon=\lambda=0}$, denoted by $L_0(\alpha)$, as $h \to \alpha$, then for h near α one can find that from Han [2]

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$$\tilde{M}(h,\delta) = \sum_{j\geqslant 0} \tilde{b}_j(\delta)(h-\alpha)^{j+1}. \tag{1.8}$$

In this case, the number of limit cycles appearing near the center $L_0(\alpha)$ can be studied by using the coefficients $\{\tilde{b}_j\}$ following the idea of [2]. For example, if there exist $k \geqslant 1$ and δ_0 such that

$$\tilde{b}_k(\delta_0) \neq 0, \quad \tilde{b}_j(\delta_0) = 0, \quad \mathrm{rank}\frac{\partial(\tilde{b}_0,\ldots,\tilde{b}_{k-1})}{\partial\delta}(\delta_0) = k, \quad j = 0,\ldots,k-1,$$

then Eq. (1.1) has at least k limit cycles for some $(\epsilon,\lambda,\delta)$ near $(0,0,\delta_0)$ at the center L_α.

This paper is devoted to giving the formulas for M_1 and M_2 appearing in (1.5), and studying the number of limit cycles of Eq. (1.1) by using M_1 and M_2. For the purpose, we write the functions H, p and q into the expansions

$$\begin{aligned} H(x,y,\lambda) &= \sum_{j=0}^{2} \lambda^j H_j(x,y) + O(\lambda^3), \\ p(x,y,0,\lambda) &= \sum_{j=0}^{2} \lambda^j p_j(x,y) + O(\lambda^3), \\ q(x,y,0,\lambda) &= \sum_{j=0}^{2} \lambda^j q_j(x,y) + O(\lambda^3). \end{aligned} \tag{1.9}$$

Then, we have the following main result.

Theorem 1.1. *Let (1.9) hold. Then*

(i) *M_1 in (1.5) has the expression*

$$M_1(h) = -\oint_{L_0(h)} H_1(x,y)\left[(p_0)_x + (q_0)_y\right]dt + \oint_{L_0(h)} q_1dx - p_1dy. \tag{1.10}$$

(ii) *Suppose that there exist a region G and C^∞ functions $p^*(x,y)$ and $q^*(x,y)$ defined on G such that for all $(x,y)\in G$*

$$-H_1(x,y)[(p_0)_x + (q_0)_y] = (H_0)_x p^* + (H_0)_y q^*.$$

Then for M_2 in (1.5) we have

$$M_2(h) = -\frac{1}{2}\oint_{L_0(h)} \varphi_1(x,y)dt - \oint_{L_0(h)} \varphi_2(x,y)dt + \oint_{L_0(h)} q_2dx - p_2dy, \tag{1.11}$$

where $L_0(h) \subset G$ and

$$\varphi_1 = (H_1p^*)_x + (H_1q^*)_y,$$
$$\varphi_2 = H_1[(p_1)_x + (q_1)_y] + H_2[(p_0)_x + (q_0)_y].$$

Consider the polynomial system

$$\begin{aligned} \dot{x} &= -y + yx^2 + \varepsilon p(x,y), \\ \dot{y} &= x + xy^2 + \varepsilon q(x,y), \end{aligned} \tag{1.12}$$

where ε is a small parameter, p and q are polynomials of degree n. Li et al. [4] studied this system by using the first Melnikov function. When perturbed Eq. (1.12) with polynomial perturbation of degree n, they found at most $0,\ 1,\ 4,\ 2\left[\frac{n+1}{2}\right]$ limit cycles for $n=0$, $n=1,2$, $n=3,4$, $n\geqslant 5$ respectively up to first order in ε. Li and Zhao [5] considered (1.12) and proved that eight limit cycles can appear for $n=3$ by applying the averaging theory of second order. In this paper, we obtain more limit cycles for the system by the above method using M_2. In other words, we have

Theorem 1.2. *For $n=3$, Eq. (1.12) can have nine limit cycles, eight of which are near the elementary center (0,0) of the reversible isochronous system $\dot{x} = -y + yx^2$, $\dot{y} = x + xy^2$.*

This paper is organized as follows. In Section 2, we derive the formulas of M_1 and M_2, which gives the proof of Theorem 1.1. In section, we will present the proof of Theorem 1.2 by using the established method.

2. Proof of Theorem 1.1

To deduce formulas for M_1 and M_2 in Eq. (1.5), we first prove the following lemma.

Lemma 2.1. *Let*

$$\bar{M}(h,\lambda) = \oint_{L_\lambda(h)} \bar{q}dx - \bar{p}dy, \tag{2.1}$$

where $\bar{p}(x,y)$ and $\bar{q}(x,y)$ are C^∞ functions independent of λ. Then

$$\bar{M}_\lambda(h,\lambda) = -\oint_{L_\lambda(h)} H_\lambda(\bar{p}_x + \bar{q}_y)dt. \tag{2.2}$$

Proof. Introduce

$$\tilde{q}(x,y) = \bar{q}(x,y) + \int_0^y \bar{p}_x(x,v)dv$$

which satisfies $\tilde{q}_y = \bar{p}_x + \bar{q}_y$. Then it follows from (2.1) and Green's formula that

$$\bar{M}(h,\lambda) = \oint_{L_\lambda(h)} \tilde{q}dx.$$

Let $A(h,\lambda) = (a(h,\lambda),\tilde{a}(h,\lambda))$ and $B(h,\lambda) = (b(h,\lambda),\tilde{b}(h,\lambda))$ denote the most left and most right points of the orbit $L_\lambda(h)$. For definiteness, we can assume that $L_\lambda(h)$ has a clockwise orientation and it can be represented as $y = y_1(x,h,\lambda)$ and $y = y_2(x,h,\lambda)$ for $a(h,\lambda) \leqslant x \leqslant b(h,\lambda)$, where $y_2(x,h,\lambda) \leqslant y_1(x,h,\lambda)$. See Fig. 2.1.

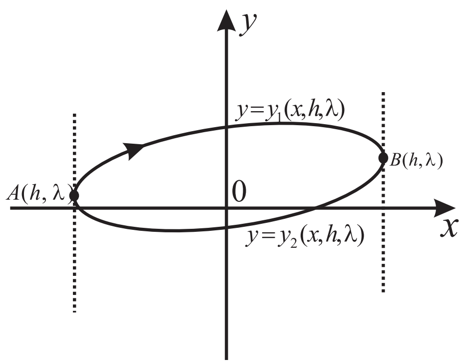

Fig. 2.1. The periodic orbit $L_\lambda(h)$.