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Low-amplitude instability as a premise for the spontaneous symmetry breaking in the new integrable semidiscrete nonlinear system

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ABSTRACT

The new integrable semidiscrete multicomponent nonlinear system characterized by two coupling parameters is presented. Relying upon the lowest local conservation laws the concise form of the system is given and its selfconsistent symmetric parametrization in terms of four independent field variables is found. The comprehensive analysis of quartic dispersion equation for the system low-amplitude excitations is made. The criteria distinguishing the domains of stability and instability of low-amplitude excitations are formulated and a collection of qualitatively distinct realizations of a dispersion law are graphically presented. The loop-like structure of a low-amplitude dispersion law of reduced system emerging within certain windows of adjustable coupling parameter turns out to resemble the loop-like structure of a dispersion law typical of beam-plasma oscillations. Basing on the peculiarities of low-amplitude dispersion law as the function of adjustable coupling parameter it is possible to predict the windows of spontaneous symmetry breaking even without an explicit knowledge of the system Lagrangian function. Having been rewritten in terms of properly chosen modified field variables the reduced four wave integrable system can be qualified as consisting of two coupled nonlinear lattice subsystems, namely the self-dual ladder network and the vibrational ones.

1. Introduction

Since the discovery of first integrable nonlinear dynamical models on a regular one-dimensional lattice [1–4] the interest to the development of new integrable semidiscrete nonlinear systems has been steadily supported by the wide range of physical problems, where the spatial discreteness and regularity play a crucial role. Among the most typical physical objects, where the semidiscrete nonlinear systems found their applications, are the optical waveguide arrays [5], semiconductor superlattices [6,7], electric superstructures [8] as well as the regular macromolecular structures of both natural [9] and synthetic [10] origin.

Evidently the more complex nonlinear physical phenomenon requires the more rich nonlinear model for its adequate description. The richness of semidiscrete integrable nonlinear system is dictated by the order of auxiliary spectral matrix $L(n|z)$ consistent with some evolution matrix $A(n|z)$ in the framework of system zero-curvature representation

$$\dot{L}(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z). \tag{1.1}$$

Here the dot written over the matrix $L(n|z)$ in the left-hand side of zero-curvature equation (1.1) means the differentiation with respect to time τ, the integer n denotes the discrete spatial coordinate running from minus to plus infinity, while z denotes the auxiliary spectral parameter independent of time: $\dot{z} = 0$.

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According to Caudrey [11,12] the order of spectral matrix $L(n|z)$ is determined by the number of *distinct* eigenvalues of either of limiting spectral matrices $L^-(z) = \lim_{n\to-\infty} L(n|z)$ or $L^+(z) = \lim_{n\to+\infty} L(n|z)$. The order of $L(n|z)$ depends both on the rank of limiting spectral matrix and on its matrix structure. Here it is worth noticing that the auxiliary spectral problems linked with known multicomponent semidiscrete nonlinear Schrödinger systems [13–16] taken at vanishing boundary conditions must be treated as the second-order ones despite being rather sophisticated matrix generalizations of the basic Ablowitz–Ladik spectral problem [3,4]. The similar property is typical of the auxiliary spectral problems associated with the matrix generalizations [17,18] of nonlinear Toda system [1,2]. From the physical standpoint the generalized systems in both of just mentioned examples do not acquire the brand-new physical quality inasmuch as the number of parameters responsible for the nonlinear couplings remains the same as in their prototype twins. In the case of semidiscrete nonlinear Schrödinger systems this statement can be confirmed by the direct consideration of low-amplitude normal modes exhibiting essentially the same dependences on wave vector both in the prototype and generalized systems despite the effect of parallel splitting admissible in the latter ones.

Meanwhile recently we have suggested early unknown semidiscrete integrable nonlinear systems [19,20] associated with the fourth order spectral problem, whose spectral matrix reads as follows

$$L(n|z) = \begin{pmatrix} 0 & t_{12}(n) & u_{13}(n)z^{-1} & 0 \\ t_{21}(n) & r_{22}(n)z^2 + t_{22}(n) & s_{23}(n)z + u_{23}(n)z^{-1} & s_{24}(n)z \\ u_{31}(n)z^{-1} & s_{32}(n)z + u_{32}(n)z^{-1} & t_{33}(n) + v_{33}(n)z^{-2} & t_{34}(n) \\ 0 & s_{42}(n)z & t_{43}(n) & 0 \end{pmatrix}. \tag{1.2}$$

Each of the systems is characterized by several coupling parameters and appears to have all chances to manifest the effect of spontaneous symmetry breaking (in a sense adopted in the theory of fields [21,22]) playing the fundamental role in many branches of physics. In view of the very complicated structure of above systems we have decided to verify the idea about symmetry breaking on a more simple but new and still integrable nonlinear system characterized at least by two coupling parameters.

The first step in this direction was to obtain an appropriate new semidiscrete integrable nonlinear system in the framework of zero-curvature scheme seeking the auxiliary spectral matrix as the third-order one. In so doing it was reasonable to keep some elements of succession between the antecendent [19,20] and sought-for schemes otherwise the procedure of empirical selection of auxiliary spectral matrix consistent with a proper auxiliary evolution matrix in the framework of zero-curvature approach may fail to be fruitful (see expressions (1.2) and (2.2) for the previous $L(n|z)$ and new $M(n|z)$ spectral operators for comparison). The above observation, when combined with the Caudrey definition of the order of a spectral operator [11,12,23], has allowed us to reveal the constructive version of early unknown third-order auxiliary spectral matrix (2.2) giving rise to new integrable systems.

Having been restricted to the reduced semidiscrete nonlinear integrable system in symmetric parametrization we have carried out the comprehensive analysis of its low-amplitude excitations under assumption of real-valued adjustable coupling parameter. Namely, the linear analysis constitutes the second step of our investigation allowing to detect the windows of spontaneous symmetry breaking in each of two reduced nonlinear system under study. The approach does not operate with the system Lagrangian function whose sole isolation seems to be an essentially nontrivial task. The same linear analysis is expected to be helpful in detecting all qualitatively distinct regimes of nonlinear (soliton) dynamics predetermined by the distinct intervals of adjustable coupling parameter framed by the critical points.

2. Zero-curvature equation and mutually consistent auxiliary matrices

In order to ensure the integrability of desired nonlinear system one need to approbate the zero-curvature equation [24]

$$\dot{M}(n|z) = B(n+1|z)M(n|z) - M(n|z)B(n|z) \tag{2.1}$$

by the spectral $M(n|z)$ and evolution $B(n|z)$ matrices chosen properly among the square matrices assumed as Laurent polynomials of spectral parameter z.

The arguments given in Introduction prompt us to define the spectral matrix $M(n|z)$ as the following 3×3 matrix

$$M(n|z) = \begin{pmatrix} z^2 + T(n) & \beta F_+(n)z + \alpha F_+(n) & G_+(n)z + G_-(n)z^{-1} \\ \alpha F_-(n)z + \beta F_-(n) & 0 & \alpha F_-(n) + \beta F_-(n)z^{-1} \\ G_-(n)z + G_+(n)z^{-1} & \beta F_+(n) + \alpha F_+(n)z^{-1} & T(n) + z^{-2} \end{pmatrix} \tag{2.2}$$

and to seek the evolution matrix $B(n|z)$ in the form

$$B(n|z) = \begin{pmatrix} a(n)z^2 + d(n) & \beta b_+(n)z + \alpha b_+(n) & c_+(n)z + c_-(n)z^{-1} \\ \alpha b_-(n)z + \beta b_-(n) & d(n) - c(n) & \alpha b_-(n) + \beta b_-(n)z^{-1} \\ c_-(n)z + c_+(n)z^{-1} & \beta b_+(n) + \alpha b_+(n)z^{-1} & d(n) + a(n)z^{-2} \end{pmatrix}, \tag{2.3}$$

where α and β are some fitting parameters independent of time. Then the direct calculations based on the zero-curvature equation (2.1) confirm our conjecture and permit to decipher almost all matrix elements $B_{jk}(n|z)$ of the tested evolution matrix $B(n|z)$ through the matrix elements $M_{jk}(n|z)$ of chosen spectral matrix $M(n|z)$ provided

$$\alpha^2 + \beta^2 = 0. \tag{2.4}$$

Thus, for the functions entering into the evolution matrix $B(n|z)$ we have

$$a(n) = k, \tag{2.5}$$
$$b_+(n) = kF_+(n), \tag{2.6}$$
$$b_-(n) = kF_-(n-1), \tag{2.7}$$
$$c_+(n) = kG_+(n), \tag{2.8}$$
$$c_-(n) = kG_-(n-1), \tag{2.9}$$
$$d(n) = -k\alpha\beta F_+(n)F_-(n-1) - kG_+(n)G_-(n-1), \tag{2.10}$$

where the summation quantity k can be thought as an arbitrary function of time τ. The only exception is the