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Infinitely many new solutions and the closed form of the solution for initial-value problem of the Burgers equation

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Abstract

Infinitely many simple-solitary-wave solutions and infinitely many rational function solutions, especially the Hopf–Cole's transformation and the closed form of the solution for initial-value problem of the Burgers equation, are obtained by using the extended homogeneous balance method. The method which is used here is simple and can be generalized to deal with other classes of nonlinear equations.

1. Introduction

Nonlinear partial differential equations (NLPDEs) are known to describe a wide variety of phenomena not only in physics, where applications extend over magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas, just to name a few, but also in biology and chemistry, and several other fields. It is one of the important tasks in the study of the NLPDEs to seek exact and explicit solutions. In the past several decades both mathematicians and physicists have made many attempts in this direction. Various methods for obtaining exact solutions to NLPDEs had been proposed. Among these are the inverse scattering transform method, Bäcklund transformation method, Darboux transformation method, Hirota's bilinear method, etc. All these methods are described in Refs. [1–4]. Recently, Wang [5] presented a homogeneous balance method (HBM) for finding exact solutions of a given NLPDEs. This method provides a convenient analytical technique to construct solitary-wave solutions and has been generalized to obtain multiple soliton (or multiple solitary-wave) solutions [6–8]. Moreover, we combine the HBM with the variable separation approach (VSA) in Ref. [9] and obtain many new types of localized excitations for some high dimensions systems. In this paper, HBM for solving NLPDEs is further extended to solving initial-value problem and deriving new solution(s) from a known solution of the equation(s) under consideration. As an illustrative example, we shall consider the Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1}$$

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which has been used to model turbulent flow in a channel and the structure of a shock wave [10]. By using the extended homogeneous balance method (EHBM), we obtain infinitely many simple-solitary-wave solutions and infinitely many rational function solutions, especially the Hopf–Cole's transformation and the closed form of the solution for initial-value problem of the Burgers equation (1).

The present article falls into four parts. In Section 2, the general theory of the EHBM is described. In Section 3, the Burgers equation is chosen to illustrate the method. Infinitely many simple-solitary-wave solutions and infinitely many rational function solutions, especially the closed form of the solution for initial-value problem of the Burgers equation, are obtained by using the EHBM given in Section 2. A brief discussion and summary is given in the last section.

2. General theory of extended homogeneous balance method

Before we describe what is EHBM, we re-describe here the HBM introduced in Ref. [5] in the following way:

Let us have a nonlinear partial differential equation, say, in two variables:

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{2}$$

where P is in general a polynomial function of its arguments, the subscripts denote the partial derivatives. Function $\varphi(x,t)$ is called a quasi-solution of Eq. (2), if there exists a function $f = f(\varphi)$ of only one variable so that a suitable linear combination of the following functions:

$$1, f(\varphi), f(\varphi)_x, f(\varphi)_t, f(\varphi)_{xx}, f(\varphi)_{xt}, f(\varphi)_{tt}, \ldots \tag{3}$$

is actually a solution of Eq. (2). The HBM for finding $f(\varphi)$, the quasi-solution $\varphi(x,t)$, the suitable linear combination of the functions in Eq. (3) and then obtaining special exact solutions of Eq. (2) consists of four steps:

First step: choosing a suitable linear combination of the functions in Eq. (3), with its coefficients to be determined so that the highest nonlinear terms and the highest-order partial derivatives terms in the given equation are both transformed into the polynomials with a highest degree in partial derivatives of $\varphi(x,t)$ with respect to $f(\varphi)$ and its various derivatives.

Second step: substituting the combination chosen in the first step into Eq. (2), collecting all terms with the highest degree of derivatives of $\varphi(x,t)$ and setting its coefficient to zero, we obtain an ordinary differential equation (ODE) for $f(\varphi)$ and then solve it; in most cases $f(\varphi)$ is a logarithmic function.

Third step: starting from the ODE for $f(\varphi)$ and its solution obtained above, the nonlinear terms of various derivatives of $f(\varphi)$ in the expression obtained in the second step can be replaced by the corresponding higher-order derivatives of $f(\varphi)$. After doing this, collecting all terms with the same order derivatives of $f(\varphi)$ and setting the coefficient of each order derivative of $f(\varphi)$ to zero, we obtain a set of equations for $\varphi(x,t)$ and combination coefficients; the left-hand sides of these equations are the k-degree homogeneous functions in various derivatives of $\varphi(x,t)$, where k is the order of $f^{(k)}$. If there exists a solution for these equations of $\varphi(x,t)$ and combination coefficients, the combination chosen in the first step can be determined.

Fourth step: substituting $f(\varphi)$ and $\varphi(x,t)$, as well as some constants obtained in the second and third steps into the combination chosen in the first step, after doing some calculations, we then obtain an exact solution of Eq. (2).

In order to obtain new solutions of Eq. (2), we add a known solution $u_0(x,t)$ to the combination obtained in the first step above. The second step is the same as that one above. In the third step, we obtain a set of equations of $\varphi(x,t)$, some coefficients of which may contain $u_0(x,t)$ or/and its derivatives. If the set of equations obtained in the third step is solvable, we solve them and get a new quasi-solution $\varphi(x,t)$. Then, proceeding as in the fourth step above, we may obtain new solution(s) of Eq. (2). In next section, we shall use EHBM obtained here to construct some new solutions of the system (1).

3. New exact solutions and the closed form of the solution for initial-value problem of the Burgers equation

According to the EHBM, in order that the nonlinear term uu_x and the second order derivative term $-\nu u_{xx}$ in Eq. (1) can be partially balanced, we suppose that the solution of Eq. (1) is of the form

$$u(x,t) = q(x,t) + \alpha f(\phi(x,t))\phi_x, \tag{4}$$

where the function $f(\phi(x,t))$ and $\phi(x,t)$ as well as the constant α are to be determined later, the function $q(x,t)$ is a known solution of Eq. (1). Substituting Eq. (4) into Eq. (1), we obtain