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Blinking fractals and their quantitative analysis using infinite and infinitesimal numbers

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Abstract

The paper considers a new type of objects – blinking fractals – that are not covered by traditional theories studying dynamics of self-similarity processes. It is shown that the new approach allows one to give various quantitative characteristics of the newly introduced and traditional fractals using infinite and infinitesimal numbers proposed recently. In this connection, the problem of the mathematical modelling of continuity is discussed in detail. A strong advantage of the introduced computational paradigm consists of its well-marked numerical character and its own instrument – Infinity Computer – able to execute operations with infinite and infinitesimal numbers.

1. Introduction

Fractal objects have been very well studied during last few decades (see, e.g., [8,18] and references given therein) and have been applied in various fields (see numerous applications given in [4,7,8,10,18,27]). However, mathematical analysis of fractals (except, of course, a very well developed theory of fractal dimensions) very often continues to have mainly a qualitative character and tools for a quantitative analysis of fractals at infinity are not very rich yet. Nowadays, the necessity of introduction of such tools becomes very urgent in connection with the appearance of new powerful approaches modelling the spacetime by fractals (see [5–7,12,13,26] and references given therein).

In this paper, we propose to apply a recently developed methodology using explicitly expressible infinite and infinitesimal numbers for two purposes: on the one hand, for a quantitative analysis of traditional and newly introduced blinking fractals; on the other hand, for developing new mathematical tools better describing (in comparison with traditional mathematical instruments developed for this goal) physical notions of continuity and discontinuity.

Let us start by introducing the new class of objects – blinking fractals – that are not covered by traditional theories studying self-similarity processes. Traditional fractals are constructed using the principle of self-similarity that infinitely many times repeats a basic object (some times slightly modified in time). However, there exist processes and objects that

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evidently are very similar to classical fractals but cannot be covered by the traditional approaches because several self-similarity mechanisms participate in the process of their construction. Before going to a general definition of blinking fractals let us give just three examples of them.

The first example is derived from one of the famous fractal constructions – the coast of Britain – as follows. Suppose that we have made a picture of the coast two times using the same scale of the map: at the moment of the early sunrise and at the moment of late sunset. Then, due to the long shadows present at these moments and directed to the opposite sides we shall have two different pictures. If we suppose, for example, that sunset corresponds to shadows on the left and sunrise to shadows on the right, then we can indicate them as L and R, correspondingly. If now we start to make pictures (starting from sunrise) alternating moments of the photographing from sunrise to sunset and decreasing the scale each time, we shall obtain a series of pictures being very similar to traditional fractals but different because left shadows will alternate right shadows at this sequence as follows: $R, L, R, L, R, L, \ldots$ Thus, there are two fractal mechanisms working in our process. Each of them can be represented by one of subsequences $R, R, R, \ldots$ and $L, L, L, \ldots$ and the traditional analysis does not allow us to say what will be the limit fractal object and will it have L or R type of shadow.

The second example is constructed as follows. Let us take a prism (see Fig. 1) that is rotating around its vertical axis and observe it at two different moments. The first is the moment when we see its face being the blue rectangular with sides 1 and $\sqrt{2}$. Since we look exactly at the front of the prism we see the rectangular as the square with the length one on side. The second moment is when we look at the face being the red right isosceles triangle with the legs equal to one. Then we apply to this three-dimensional object the two following self-similarity rules: we substitute each prism by four smaller prisms during the time passing between each even and odd observation and by two smaller prisms during the time passing between each odd and even observation. Thus, at the odd iterations we observe application of the first mechanism shown in the top of Fig. 2. The second mechanism shown in the bottom of this figure is applied during the even iterations. As a result, starting from the blue square one on side at iteration 0 we observe the pictures (see Fig. 3) with alternating blue squares and red triangles. Again, as it was with the above example related to the coast of Britain, we can extract two fractal subsequences being traditional fractals. The mechanism of the first one dealing with blue squares is shown in Fig. 4. The second mechanism dealing with red triangles is presented in Fig. 5. Traditional approaches are not able to say anything about the behavior of this process at infinity. Does there exist a limit object of this process? If it exists, what can we say about its structure? Does it consist of red triangles or blue squares? What is the area of this (again, if it exists) limit object? All these questions remain without answers.

Before we discuss the last example linked, as it was with our first example, to another famous fractal construction – Cantor's set (see Fig. 6) – let us make a few comments reminding that very often we can give certain numerical answers to questions regarding fractals only if a finite number of steps in the procedure of their construction has been considered. The same questions very often remain without any answer if we consider an infinite number of steps. If a finite

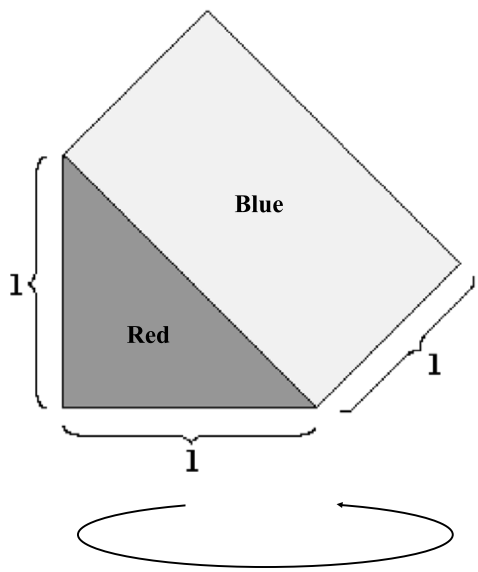

Fig. 1. The rotating prism having the triangular face red and the rectangular face blue.