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Kink-soliton explosions in generalized Klein–Gordon equations

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Abstract

We investigate the dynamics of kink-solitons in generalized Klein–Gordon equations in the presence of nonlinear damping and spatiotemporal perturbations. We present different mechanisms for kink-soliton explosion and show that, in some cases while some analytically obtained conditions are satisfied, the kink-soliton breaks up becoming a permanent, extremely complex, spatiotemporal dynamics. We perform computer simulations in order to visualize and corroborate our analytical findings. The mechanisms presented for kink-soliton breakup can explain some of the phenomena that recently have been reported to occur in excitable media. Finally, we will present a way to control soliton breakup, using appropriate external perturbations.

1. Introduction

Solitons are self-localized solutions of nonlinear evolution equations [1–3]. One of the most attractive characteristics of solitons is that they can propagate without visible changes. However, under some conditions, solitons may become unstable [4–16]. Such instabilities had been called soliton breakup or soliton explosion.

Milchev and coworkers have studied the Frenkel–Kontorova model with anharmonic interatomic interaction [4–8] and found that a breakup of the kink takes place when the effective potential amplitude exceeds a certain critical value. An extensive discussion of soliton dynamics in the Frenkel–Kontorova model can be found in the recent book [17]. On the other hand, in Ref. [13,14] it was predicted that instability in the soliton internal mode leads to soliton breakup.

Kink-solitons [1–3,18], like vortices and spirals [19–21], are particular cases of a more general phenomenon called topological defects. Although these objects may possess different origin and nature in different physical systems, they all possess very similar dynamical properties.

The study of spiral breakup has been a very active area of research in recent years [20]. Spiral breakup has been observed in experiments and simulations [22–30].

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One possible mechanism which is currently believed to be responsible for the transition from tachycardia to ventricular fibrilation is the spontaneous breakup of a single spiral wave of electrical activity into multiple spirals leading to a turbulent wave behavior [27].

The investigation of the transition from regular defect motion to highly nonstationary spatiotemporal dynamics remains a challenge in modern science [31]. This phenomenon is related to defect-mediated turbulence [31,32].

We will present different mechanisms for soliton breakup and explosions. We find that in some cases, while some conditions hold, the soliton explosion is permanent. That is, if the conditions for soliton explosions have not changed, the soliton does not return to the original steady state. In some cases, this dynamics is very similar to defect turbulence.

In the present paper, we investigate generalized Klein–Gordon equations as the following

$$\phi_{tt} + R(\phi_t) - \phi_{xx} - G(\phi) = F(x,t), \tag{1}$$

where $G(\phi) = -\partial U(\phi)/\partial\phi$, $U(\phi)$ is a potential function with at least two minima ϕ_1, ϕ_3 and a maximum ϕ_2, such that $U(\phi_1) = U(\phi_3)$, $R(\phi_t)$ are dissipative terms, and $F(x,t)$ represents external perturbations. We are interested in kinks, that is, topological solitons between the points ϕ_1 and ϕ_3. The famous sine-Gordon and ϕ^4-systems are particular cases of Eq. (1).

The topological solitons studied in the present paper possess important applications in condensed matter physics. For instance, in solid state physics, they describe domain walls in ferromagnets and ferroelectric materials, dislocations in crystals, charge-density waves, interphase boundaries in metal alloys, fluxons in long Josephson junctions and Josephson transmission lines, etc. [1].

Such instabilities as soliton breakups can affect all mentioned applications. Therefore it is very important to understand all the possible mechanisms of soliton breakup in order to avoid and control [33] them.

In the following, we present the stability of kink-soliton solutions in Section 2, different mechanisms for soliton breakup in Section 3, and show that in some cases, while some conditions hold, the soliton becomes a permanent, extremely complex, nonstationary spatiotemporal dynamics. In Section 4, we present a way to control such breakups from happening, Conclusions are given at the end of the paper.

2. Stability of kink-solitons

It is well known that the existence of heteroclinic trajectories joining the fixed points of the corresponding dynamical system [34], is a condition for the existence of kink-soliton solutions. Therefore we perform a complete investigation of equations of type (1) using the so-called qualitative theory of dynamical systems [35,36] (including topological concepts), and with the additional information about the behavior of solutions in the neighborhood of fixed points and separatrices, it is possible to construct functions with the general properties of the exact solutions of the given equation. Then, solving an inverse problem, we are able to find exact solution for systems of type (1). Later, it is possible to generalize the results to some classes of equations that are topologically equivalent to those with the exact solutions.

In some cases, it is sufficient to have an exact solution corresponding to some relevant physical situation and for which we can solve exactly the stability problem.

For instance, let us study the following equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = F(x). \tag{2}$$

We are interested in the stability of kinks at the equilibrium positions created by the inhomogeneous force $F(x)$. Constructing a general function $\phi_k(x)$ with the topological and general properties of a kink, we solve an inverse problem such that $F(x)$ possesses the properties of the physical system we are studying. Now we can investigate the stability of the solution $\phi(x,t) = \phi_k(x) + f(x)e^{\lambda t}$. For this we solve the spectral problem

$$\widehat{L}f(x) = \Gamma f(x), \tag{3}$$

where

$$\widehat{L} = -\partial_{xx} - \left(\frac{\partial G(\phi)}{\partial\phi}\right)_{\phi=\phi_k(x)},$$

$$\Gamma = -(\lambda^2 + \gamma\lambda).$$

The results obtained with this function can be generalized to other systems topologically equivalent to the exactly solvable system [9–12,18,37–39].