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Chaos, Solitons and Fractals 33 (2007) 156–162
CHAOS
SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Limit cycles design for a class of bilinear control systems

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Accepted 28 December 2005

Abstract

In this paper, the feedback control for a class of bilinear control systems is investigated. Using the Bellman–Gronwall inequality, a feedback control is proposed to guarantee the existence of limit cycles for such bilinear control systems. Moreover, the exponentially stable limit cycles, the guaranteed convergence rate, and frequency of oscillation can be correctly estimated. Finally, a numerical example is provided to illustrate the use of the main result.

1. Introduction

Nonlinear network can offer oscillations with fixed amplitude and fixed period. These oscillations are named limit cycles, e.g., Van der Pol equation and an RLC electrical circuit with a nonlinear resistor. Limit cycles are unique phenomenon of nonlinear networks and have been a main concern of the researchers over the years; see, for instance, [1–8], and the references therein. Prediction of limit cycles is very significant, because limit cycles can happen in any kind of physical system. Frequently, a limit cycle can be desirable. This is the case of limit cycles in the electronic oscillators utilized in laboratories. There are at least four methodologies to investigate the phenomenon of limit cycles, namely describing function method, Poincare–Bendixson theorem, Piecewise-linearized methods [7], and Lyapunov-like approach [8]. The drawbacks of the describing function method are related to its approximate nature, and include the possibility of incorrect predictions. Furthermore, the Poincare–Bendixson theorem only provides a necessary condition to guarantee the existence of limit cycles [8]. Hence, even the conditions of the Poincare–Bendixson theorem are satisfied for some system, the existence of limit cycles cannot be guaranteed for such system.

In this paper, using the Bellman–Gronwall inequality, a feedback control is proposed to guarantee the existence of limit cycles for a class of bilinear control systems. In addition, the exponentially stable limit cycles and frequency of oscillation can be correctly estimated. Finally, an estimate of the guaranteed convergence rate is derived for such bilinear control systems.

2. Problem formulation and main results

In this paper, we consider the following bilinear control systems [9]:

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2006.01.004

$$\dot{x}(t) = Ax(t) + u(t)Nx(t), \quad \forall t \geqslant t_0 \geqslant 0, \tag{1a}$$

$$x(t_0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \tag{1b}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Re^2, \quad u \in \Re, \quad A := \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad N := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

with $a > 0$ and $x(t_0) \neq 0$. Specially, the feedback control law is selected as follows:

$$u(t) = \bar{r}(x_1^2(t) + x_2^2(t)), \quad \text{with } \bar{r} > 0. \tag{1c}$$

Thus, the closed-loop systems are deduced as

$$\begin{cases} \dot{x}_1(t) = -bx_2(t) - \bar{r}x_1(t)\left[x_1^2(t) + x_2^2(t) - \dfrac{a}{\bar{r}}\right], \\ \dot{x}_2(t) = bx_1(t) - \bar{r}x_2(t)\left[x_1^2(t) + x_2^2(t) - \dfrac{a}{\bar{r}}\right], \quad \forall t \geqslant t_0. \end{cases} \tag{2}$$

Obviously, $x = 0$ is an equivalent point of system (2), i.e., the solution of system (2) is given by $x(t) = 0$ if $x(t_0) = 0$. To avoid the trivial case of $x(t_0) = 0$, in the following, we only consider the system (1) under the case of $x(t_0) \neq 0$.

Definition 1 [8]. Consider the system (2). The closed and bounded manifold $s(x) = 0$, in the x_1–x_2 plane, is said to be an exponentially stable limit cycle if there exist two positive numbers α and β such that the manifold of $s(x) = 0$ along the trajectories of system (2) satisfies the following inequality:

$$|s(x(t))| \leqslant \beta \cdot \exp[-\alpha(t - t_0)], \quad \forall t \geqslant t_0.$$

In this case, the positive number α is called the guaranteed convergence rate.

Now, we present the main result for the existence of limit cycles of system (1) as follows.

Theorem 1. *For the feedback bilinear systems (1), all of phase trajectories tend to the exponentially stable limit cycle $s(x) = x_1^2 + x_2^2 - \frac{a}{\bar{r}} = 0$ in the x_1–x_2 plane, with the guaranteed convergence rate*

$$\alpha := \begin{cases} \infty & \text{if } x_{10}^2 + x_{20}^2 = a/\bar{r}, \\ 2a & \text{if } x_{10}^2 + x_{20}^2 > a/\bar{r}, \\ 2\bar{r}(x_{10}^2 + x_{20}^2) & \text{if } x_{10}^2 + x_{20}^2 < a/\bar{r}. \end{cases}$$

Furthermore, the states $x_1(t)$ and $x_2(t)$ exponentially track, respectively, the trajectories

$$\sqrt{\frac{a}{\bar{r}}}\cos\left[b(t - t_0) + \tan^{-1}\left(\frac{x_{20}}{x_{10}}\right)\right] \quad \text{and} \quad \sqrt{\frac{a}{\bar{r}}}\sin\left[b(t - t_0) + \tan^{-1}\left(\frac{x_{20}}{x_{10}}\right)\right],$$

in the time domain, with the guaranteed convergence rate $\alpha/2$.

Proof. Define a smooth manifold $s(x) = 0$ and a continuous function $\theta(x) := \tan^{-1}\left[\frac{x_2}{x_1}\right]$ with $s(x) = x^T x - \frac{a}{\bar{r}}$. Then the time derivatives of $s^2(x)$ and $\theta(x)$ along the trajectories of system (2) is given by

$$\begin{aligned} \frac{\mathrm{d}s^2(x(t))}{\mathrm{d}t} &= 2s(x(t)) \cdot (2x_1\dot{x}_1 + 2x_2\dot{x}_2) = -4\bar{r}(x_1^2 + x_2^2)s^2(x(t)), \\ \frac{\mathrm{d}\theta(x(t))}{\mathrm{d}t} &= \frac{\dot{x}_2x_1 - \dot{x}_1x_2}{x_1^2 + x_2^2} = b, \end{aligned} \tag{3}$$

which imply

$$\theta(x(t)) = b(t - t_0) + \tan^{-1}\left(\frac{x_{20}}{x_{10}}\right). \tag{4}$$

In the following, there are three cases to discuss the trajectories of the feedback control system of (2).

Case 1: $x_1^2(t_0) + x_2^2(t_0) = a/\bar{r}$ (or equivalently; $s(x(t_0)) = 0$)

In this case, from (3), it can be obtained that $\frac{\mathrm{d}s^2(x(t))}{\mathrm{d}t} = 0$, which implies

$$x_1^2(t) + x_2^2(t) = a/\bar{r}, \quad \forall t \geqslant t_0. \tag{5}$$