

Topological horseshoes in continuous maps

Xiao-Song Yang

Department of Mathematics, Huazhong University of Science and Technology, 430074 Wuhan, China

Accepted 13 December 2005

Abstract

We present a framework on existence of horseshoes in continuous maps in terms of “crossing with respect to two pairs of fixed subsets”.

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1. Introduction

The existence of a “horseshoe” embedded in a dynamical system should be the most compelling signature of chaos, both in dissipative and conservative systems. Now it is recognized that horseshoe theory (Smale horseshoe or current topological horseshoe theory) with symbolic dynamics provides a powerful tool in rigorous studies of complicated dynamics such as chaos in dynamical systems. Since the first result on Smale horseshoes, a remarkable progress has been made in seeking sufficient conditions for existence of horseshoes in dynamical systems [1–4,7–12], both in discrete time and continuous time (by virtue of Poincaré map) cases.

A noteworthy result along this line of research should be the topological horseshoe theory obtained by Kennedy and Yorke [1], where the theory on topological horseshoe was established in context of continuous map from a metric space to itself. The key in their theory [1] is the so-called “cross number”, by which the statement that a continuous map having “cross number” M in an invariant Q will be semi-conjugate to M shift map can be established based on the fact that having “cross number” M implies existence of M mutually disjoint sets in connection with M shift map. However, in practice the so-called “cross number” cannot be a priori known. As a matter of fact, one often has to find M mutually disjoint sets in studying chaos in concrete physical and engineering systems without any information about the “cross number”. Therefore how to find M mutually disjoint sets by which one can establish semi-conjugacy relationship of a continuous map to M shift map is of more practical significance. Along this line comes up several papers by Zgliczyński, Mischaikow, Mrozek [3,4], and Yang et al. [7–10]. Among these the most recent papers by Zgliczyński and Gidea [11,12] should be mentioned.

The present paper continues the effort of seeking conditions on existence of horseshoes in the context of M mutually (not necessarily) disjoint sets with “crossing with respect to two pairs of fixed subsets” property. The present paper was motivated both from the work by Kennedy and York [1] and the author’s experiences in investigation of some concrete dynamical systems [7,8]. The conditions proposed in the present paper for existence of chaos are easy to verify in many concrete systems and the arguments do not need knowledge of differential topology and deep set topology.

E-mail address: yangxs@cqupt.edu.cn

This paper is organized as follows. Section 2 gives some preliminaries concerning symbolic dynamics as well as some basic facts about topological entropy. Section 3 revisits a topological horseshoe lemma that plays a fundamental role in establishing sufficient conditions on existence of topological horseshoe in continuous or piecewise continuous map. Section 4 introduces the notion and notation of properly across and other related concepts. Based on the horseshoe lemma, several new theorems on existence of topological horseshoe are proved in Section 4. In Section 5, some comments are presented.

2. Preliminaries

First we recall some aspects of symbolic dynamics.

Let $S_m = \{0, 1, \dots, m-1\}$ be the set of nonnegative integers from 0 to $m-1$. Let Σ_m be the collection of all bi-infinite sequences (one-sided sequences) with their elements of S_m , i.e., every element s of Σ_m is of the following form

$$s = \{\dots, s_{-n}, \dots, s_{-1}, s_0, s_1, \dots, s_n, \dots\}, \quad s_i \in S_m$$

or

$$s = \{s_0, s_1, \dots, s_n, \dots\}, \quad s_i \in S_m.$$

Now consider another sequence $\bar{s} \in \Sigma_m$

$$\bar{s} = \{\dots, \bar{s}_{-n}, \dots, \bar{s}_{-1}, \bar{s}_0, \bar{s}_1, \dots, \bar{s}_n, \dots\}, \quad \bar{s}_i \in S_m$$

or

$$\bar{s} = \{\bar{s}_0, \bar{s}_1, \dots, \bar{s}_n, \dots\}, \quad \bar{s}_i \in S_m.$$

The distance between s and $\bar{s}$ is defined as

$$d(s, \bar{s}) = \sum_{-\infty}^{\infty} \frac{1}{2^{|i|}} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|} \quad (1)$$

in case of bi-infinite sequences, or

$$d(s, \bar{s}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|} \quad (2)$$

in case of one-sided sequences.

With the distance defined as (1) or (2), Σ_m is a metric space, and the following facts are well known [6].

Proposition 2.1. *The space Σ_m is*

- (i) *compact,*
- (ii) *totally disconnected,*
- (iii) *perfect.*

A set having the three properties in the above proposition is often defined as a Cantor set, such a Cantor set frequently appears in characterization of complex structure of invariant set in a chaotic dynamical system.

Furthermore, now define a m -shift map $\sigma: \Sigma_m \rightarrow \Sigma_m$ as follows

$$\sigma(s)_i = s_{i+1}.$$

Then there is the following result.

Proposition 2.2. (a) $\sigma(\Sigma_m) = \Sigma_m$ and σ is continuous. (b) *The shift map σ as a dynamical system defined on Σ_m has the following properties:*

- (i) σ has a countable infinity of periodic orbits consisting of orbits of all periods;
- (ii) σ has an uncountable infinity of nonperiodic orbits; and
- (iii) σ has a dense orbit.

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