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Bifurcation analysis in the diffusive Lotka–Volterra system: An application to market economy ☆

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Abstract

A diffusive Lotka–Volterra system is formulated in this paper that represents the dynamics of market share at duopoly. A case in Sri Lankan mobile telecom market was considered that conceptualized the model in interest. Detailed Hopf bifurcation, transcritical and pitchfork bifurcation analysis were performed. The distribution of roots of the characteristic equation suggests that a stable coexistence equilibrium can be achieved by increasing the innovation while minimizing competition by each competitor while regulating existing policies and introducing new ones for product differentiation and value addition. The avenue is open for future research that may use real time information in order to formulate mathematically sound tools for decision making in competitive business environments.

1. Introduction

Diffusion is the process by which an innovation is communicated through certain channels over time among the members of a social system [18]. In economic context, the goal of a diffusion model is to explain the process of an innovation spreading among individuals. The commonly used model that describe the market diffusion or the product growth forecasting is the Bass model [1] where many variants have been developed that address issues in a range of fields [8,10,16]. Furthermore, it is observed that the dynamical behavior of the general Bass model is very similar to the Lotka–Volterra (LV) population growth model with no competition [21]. However, when a competition is present, the approach in market share attraction models has been one of the better alternatives used in a variety of contexts to describe the behavior of competitors in a market (see [2] and reference therein).

The first differential equation of predator–prey type was formulated by Alfred Lotka and Vito Volterra early in the last century, when attempts were first made to find ecological laws of the nature. Since then, the study of the LV system has attracted attention of great number of investigators (see [13,14,20] and references therein). Usually, the LV equation represent a simple non-linear model for the dynamic interaction between two species in which one species benefits at the

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expense of the other. The same model can also be applied to two species that compete for resources or that symbiotically interact in the same environment for survival. In the literature on evolutionary economics, the extensive application of Lotka–Volterra equation in business and economics is prominent [4,6].

When the current mobile telecom market situation in Sri Lanka is considered, it is not hard to understand the rival competition of the two most competitive mobile services providers [17,22,26]. These competitors are yet to expanding their geographical coverage, where they penetrate the market in areas where niches are available. When they penetrate a new area, customers are promoted to subscribe to special offers and the customers within a particular package will get more benefits than what others do. This opens up more active competition among subscribers of different service providers where they are promoted to attract customers from other service providers. Therefore, it is clear fact that stronger densities of customer base in a particular area will result higher rate of diffusion and vice versa. A study conducted by Somasundara and Withanage [19] in Ratnapura district (Sabaragamuwa province of Sri Lanka) confirmed that the coverage area of the mobile telephone services should be increased while improving the quality of the service and the consumer relations. It has also been confirmed that more than 80% of the phone-less people intend to buy a telephone. This indicates that still there is a high demand, which has not yet been captured by the telecommunication service providers in the Ratnapura district. In order to conceptualize the model, authors consider a case in the Sabaragamuwa University of Sri Lanka, where it has been identified as a niche for mobile telecom market. According to a personal discussion made with the regional sale's depot, until 2005, Mobitel analog service owned the 100% share of the market where this monopolistic situation did not provide benefits of the modern mobile telephony to its customers. During late 2005, Dialog GSM started offering its service with modern GSM facilities and attractive group packages and claimed for 70% share of the mobile market in the area by 2006. Thereafter Mobitel introduced new GSM service to the area to face competition imposed by Dialog GSM where the percentage share held by the two competitors have been dramatically fluctuated with the time. The situation currently going on in Sri Lankan mobile telephone market is similar to species competition for invasion and survival, where one aggressive competitor invades an area which is occupied by another species that competes for the same resources for the survival. Therefore, better competitors in an open market situation are thriving with full market share while less competitive business may continued to be deteriorated.

When the market shares are considered they will sum to unity (or 100 as a percentage) and the individual values are in between 0 and 1 [2] (or 0 and 100 as a percentage). Therefore, the competition would result a periodic behavior in unsaturated markets where the market share would oscillate around a given threshold level where the bifurcation near the fixed points helps determine the critical and maximum shares of the market. According to [15] in the modern neoclassical economics it takes two approaches to economic fluctuations and business cycles: the endogenous approach and the exogenous approach. While the exogenous approach views fluctuations and cycles as a result of random shocks, the endogenous approach attributes them to the fundamental structure of an economy resulting in complex non-linear dynamics. Intuitively the approach in the present study focuses on endogenous approach where random shocks (chaos) within a given time limit assumed to be negligible.

Instead of using percentage values, a standardized score (Z) can be obtained for the market shares of each competitor at time t, which would allow the negative fixed points to appear in the solution structure when periodic solutions are occurred:

$$Z_{ti} = \frac{x_{ti} - \mu_i}{\sigma_i},$$

where Z_{ti} is the standardized value of the customer density (market share) of ith competitor at time t, x_{ti} is the observed customer density, μ_i and σ_i are mean and the standard deviations, respectively. According to Chebyshev's Theorem, there is no harm to compute the Z-score with any set of values where the distribution can have any shape (see [12]). Therefore, standardized values would be considered herein.

In this paper, we are interested in studying the following diffusive LV system by assuming that the geographical expansion of the market share of each competitor obeys a reaction–diffusion Eq. (1.1):

$$\begin{cases} u_t = \Delta u + u(1 - u - kv), & x \in \Omega, \ t > 0, \\ v_t = \theta\Delta v + v(\lambda - \lambda v - \delta u), & x \in \Omega, \ t > 0, \end{cases} \tag{1.1}$$

subject to the following no-flux boundary condition:

$$\partial_\nu u = \partial_\nu v = 0, \quad x \in \partial\Omega, \ t > 0, \tag{1.2}$$

where Ω is the bounded domain in R^n, with the smooth boundary $\partial\Omega$. Here $u = u(x,t)$ and $v = v(x,t)$ stand for the standardized values of customer densities at time $t > 0$ and a position $x \in \Omega$, respectively. The given coefficients k, λ, δ, θ are positive constants; Here θ is the ratio of diffusion coefficients of the competitors. So the components with the coefficients d and θ represent the process of neighborhood dependent growth i.e. the influence of the customers who are already