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The origin and proof of quantization axiom $\mathbf{p} \to \hat{\mathbf{p}} = -i\hbar\nabla$ in complex spacetime

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Abstract

The quantization axiom $\mathbf{p} \to \hat{\mathbf{p}} = -i\hbar\nabla$ is the kernel in constructing quantum-mechanical systems; however, it was proposed without proof and even till now no formal proof has been given about its origin and validity by using fundamental theory of mechanics. This paper aims to show that quantum operators have the root in complex spacetime and can be derived naturally from complex-extended Hamilton equations of motion. The derivation of quantum operators from Hamilton mechanics is coordinate-independent and thus allows us to deduce operators directly from any curved spacetime without transforming back to Cartesian space.

1. Hamilton mechanics in complex spacetime

Since the study of E-infinity Cantorian spacetime [1–3], which revealed that many quantum phenomena are natural consequences of conducting experiments in a fractal spacetime while observing and taking measurements in a classical smooth $3+1$ spacetime, it becomes increasingly evident that the nature of quantum mechanics has its root in fractal spacetime. To manifest this fact further, this paper intends to prove, from the viewpoint of fractal spacetime, two fundamental postulates of quantum mechanics:

(1) Postulate of correspondence: to any self-consistent and well-defined observables A, there corresponds an operator $\widehat{A}$.
(2) Postulate of quantization: The operator $\widehat{A}$ corresponds to the observable $A(\mathbf{r},\mathbf{p})$ can be constructed by replacing the quantities $\mathbf{r}$ and $\mathbf{p}$ in the expression for A by the assigned operators $\mathbf{r} \to \hat{\mathbf{r}} = \mathbf{r}$ and $\mathbf{p} \to \hat{\mathbf{p}} = -i\hbar\nabla$.

Quantum mechanics had been established from the above two postulates whose validity, in turn, was justified indirectly via the voluminous precise predictions of quantum mechanics. Although the two postulates work very successfully, until now we still do not know why they should work. We also do not know very clearly about the underlying

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reason that to obtain the correct operators in coordinate system other than Cartesian coordinates, it is always necessary to transform into Cartesian coordinates before putting in the operators. This paper aims to prove the two postulates by the first principle of Hamilton mechanics, to expound the reason why the postulate of quantization is only true in the Cartesian coordinates, and to demonstrate how to obtain directly quantum operators in curvilinear coordinates without transforming back to Cartesian coordinates.

The main idea of the proof is based on the complex-space formulation of non-differentiable fractal spacetime [4–6], which relates an observable defined in complex-extended Hamilton mechanics to its associated operator $\widehat{A}$ defined in quantum mechanics. Since the behavior of $A(\mathbf{r},\mathbf{p})$ must obey Hamilton equations, we thus can determine the expression and the property of $\widehat{A}$ directly from Hamilton equations of motion, regardless of the coordinate system being used. The Hamilton equations considered here are derived from a quantum Hamiltonian H, which is different from the classical one. To find out the correct quantum Hamiltonian, we first recall a classical result that for a given classical Hamiltonian $H_c(t,\mathbf{q},\mathbf{p})$, the classical Hamilton–Jacobi (H–J) equation reads

$$\frac{\partial S_c}{\partial t} + H_c(t,\mathbf{q},\mathbf{p})|_{\mathbf{p}=\nabla S_c} = 0, \tag{1}$$

where $(\mathbf{q},\mathbf{p})$ are the canonical variables and S_c is the classical action function. We may regard the classical H–J equation as the short wavelength limit of Schrödinger equation [7]:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi, \tag{2}$$

as can be seen via the following transformation:

$$\psi = \mathrm{e}^{\mathrm{i}S/\hbar}, \tag{3}$$

from which Schrödinger equation becomes

$$\frac{\partial S}{\partial t} + \left[\frac{1}{2m}(\nabla S)^2 + V\right] = \frac{\mathrm{i}\hbar}{2m}\nabla^2 S. \tag{4}$$

We recognize the quantity in brackets as the classical Hamiltonian for a single particle described in Cartesian coordinates. Eq. (4) is known as the quantum H–J equation, which reduces to the classical H–J equation (1) if the right-hand side of Eq. (4) is negligible, which means that the wavelength of the matter wave is so short that the momentum changes by a negligible fraction over a distance of wavelength [7]. If we treat Eq. (4) as the quantum-mechanical counterpart of the classical H–J equation (1), it is natural to ask what will be the corresponding quantum-mechanical counterpart of the classical Hamiltonian H_c. Rewriting Eq. (4) in a form analogous to Eq. (1):

$$\frac{\partial S}{\partial t} + H(t,\mathbf{q},\mathbf{p})|_{\mathbf{p}=\nabla S} = 0. \tag{5}$$

We obtain the desired quantum Hamiltonian H, compatible with Schrödinger equation, as

$$H = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{q}) + Q(\psi(\mathbf{q})), \tag{6}$$

where Q is known as quantum potential defined by

$$Q(\psi(\mathbf{q})) = \frac{\hbar}{2m\mathrm{i}}\nabla\cdot\mathbf{p} = \frac{\hbar}{2m\mathrm{i}}\nabla^2 S = -\frac{\hbar^2}{2m}\nabla^2\ln\psi(\mathbf{q}). \tag{7}$$

The complex quantum potential $Q(\mathbf{q})$ with $\mathbf{q}\in\mathbb{C}$ produces a non-classical structured spacetime. The transformation (3) was first introduced by Schrödinger in transforming the phase function $\phi = S/h$ governed by Fresnel's wave equation to the wavefunction ψ governed by his wave equation. There are two roles played by the wavefunction ψ in the quantum Hamiltonian H. Firstly, it determines the canonical momentum according to

$$\mathbf{p} = \nabla S = -\mathrm{i}\hbar\nabla\ln\psi. \tag{8}$$

Secondly, it generates the quantum potential Q according to Eq. (7). The equations of motion for a particle moving in the quantum state ψ are formed by applying the quantum Hamiltonian H of Eq. (6) to the Hamilton equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial\mathbf{p}}, \tag{9a}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial\mathbf{q}}. \tag{9b}$$