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Generalized Dirac equation and its symmetries

Kourosh Nozari *

Department of Physics, Faculty of Basic Sciences, University of Mazandaran, P.O. Box 47416-1467, Babolsar, Iran
Research Institute for Astronomy and Astrophysics of Maragha, P.O. Box 55134-441, Maragha, Iran

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Abstract

Existence of a minimal observable distance on the order of Planck length is an immediate consequence of the Generalized Uncertainty Principle (GUP). As a result, relativistic quantum mechanics should be generalized to incorporate such a novel feature. The resulting theory may be considered as an effective relativistic quantum mechanics for the Planck scale regime. In this paper, we find generalized Dirac equation in GUP framework and solve its eigenvalue problem for a free particle. Lagrangian of generalized Dirac field is obtained and its symmetry properties are discussed. Finally, CPT symmetries will be re-examined in the framework of generalized Dirac theory. We show that, up to first order in GUP parameter, Noether's theorem and CPT symmetries are not broken in GUP framework.

1. Introduction

The problem of reconciling Quantum Mechanics with General Relativity is one of the task of modern theoretical physics which, until now, has not yet found a consistent and satisfactory solution. One of the most interesting consequences of this unification is that in quantum gravity there exists a minimal observable distance on the order of the Planck distance, $l_{\mathrm{P}} = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33}$ cm, where G is the Newton constant. The existence of such a fundamental length is a dynamical phenomenon due to the fact that, at Planck scale, there are fluctuations of the background metric, i.e., a limit of the order of Planck length appears when quantum fluctuations of the gravitational field are taken into account. In the language of string theory one can say that a string cannot probe distances smaller than its length. The existence of minimal observable length which is motivated from string theory [1,2], loop quantum gravity [3], Non-commutative geometry [4], E-infinity theory [15,17,26] and black hole physics [9], leads to a generalization of Heisenberg uncertainty principle to incorporate gravitational induced uncertainty from very beginning. This generalized uncertainty principle can be written as

$$\Delta x \geqslant \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \tag{1}$$

* Address: Research Institute for Astronomy and Astrophysics of Maragha, P.O. Box 55134-441, Maragha, Iran.
E-mail address: knozari@umz.ac.ir

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At energy much below the Planck mass, $m_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}$ GeV$/c^2$, the extra term in (1) is irrelevant and the Heisenberg uncertainty relation is recovered, while, as we approach the Planck energy, this term becomes relevant and, as has been indicated, it is related to the existence of minimal observable length. Now the generalized commutation relation becomes,

$$[x,p] = i\hbar(1+\beta' p^2). \tag{2}$$

Note that commutator (2) does not lead to (1) [10]. In the uncertainty inequality corresponding to a given commutation relation, the right-hand side contains the expectation value of the commutator. In this case, this will produce not only the squared uncertainty of the momentum but also the squared expectation value of the momentum. With this cautionary remarks in mind, we consider (2) as our primary input.

This feature (existence of minimal observable length), constitutes a part of the motivation to study the effects of modified Heisenberg algebra on various observable. Consequences of such a gravitational uncertainty principle (GUP), have been studied extensively (see for example [11–13]). The effect of GUP on ordinary quantum mechanics has been studied by Hossenfelder et al. [11]. They have found a generalization of the Schrödinger equation within GUP framework. Brau has considered the problem of Hydrogen atom energy spectrum in the framework of minimal uncertainty relation [14]. Also Akhoury and Yao have considered the Hydrogen atom energy spectrum in the framework of GUP [15]. Nozari and Azizi have calculated the energy spectrum of a particle in an one dimensional infinite well using the generalized Schrödinger equation in maximally localized position representation and they have found momentum space wave function of the free particle in GUP [16]. Generalized Dirac equation first have been obtained by Hossenfelder et al. [11], without any further development. Recently we have found a generalization of Dirac equation to incorporate quantum gravitational effects via the generalized uncertainty principle [17]. Here we proceed some more step in this direction. We develop the theory to complete determination of corresponding Dirac spinors, its Lagrangian formalism and investigation of inherent symmetries. First, we consider the effect of GUP on definition of momentum operator and then we obtain a generalized Dirac equation, solve its eigenvalue problem and interpret the corresponding results. We will show that it is impossible to have free particle in Planck scale. This is a consequence of the background metric fluctuation in this scale. Comparison between our results and the results of ordinary relativistic quantum mechanics shows a considerable departure when one approaches extreme quantum gravity limit. In the next step, we calculate the generalized Lagrangian for Dirac field and investigate its symmetry properties. We show that up to first order in GUP parameter, Noether theorem is valid in GUP framework. Finally we investigate the validity of CPT symmetries for generalized Dirac theory. We show that up to first order in GUP parameter, these symmetries are not broken. Note that in this paper we attains the existence of GUP as our primary input and with this assumption, we are going to find some of its consequences for relativistic quantum mechanics. In which follows, we choose $\hbar = c = 1$. In this situation $\beta' = l_p^2 = \frac{1}{m_P^2}$. Latin index i, takes the values $1,2,3$ and Greek index μ takes the values $0,1,2,3$. Index 0 stands for temporal part of 4-vectors. β and γ^i's (or $\boldsymbol{\alpha}$) are usual 4×4 matrices of relativistic quantum mechanics.

2. Generalized Dirac equation

In the framework of GUP, one can generalize usual momentum operator $p_{op} = -i\frac{\partial}{\partial x}$ to the following form [14,15]:

$$p_{op}^{(GUP)} = -i\left(1-\beta'\left(\frac{\partial}{\partial x}\right)^2\right)\frac{\partial}{\partial x}. \tag{3}$$

For an arbitrary component, this generalized momentum operator can be written as

$$p_i = -i[1-\beta'(\partial_i)^2]\partial_i. \tag{4}$$

The square of this operator is

$$p^2 = p_i p^i = (-i[1-\beta'(\partial_i\partial^i)]\partial_i)(-i[1-\beta'(\partial^i\partial_i)]\partial^i), \tag{5}$$

which up to first order in β' simplifies to

$$p^2 \simeq -[\partial_i\partial^i - 2\beta'(\partial^i\partial_i)(\partial^i\partial_i)]. \tag{6}$$

Now the Dirac equation [18],

$$(\gamma^\mu p_\mu - m)\psi = -(i\gamma^\mu\partial_\mu + m)\psi = 0, \tag{7}$$

which can be written as