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Complex tunneling dynamics

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Abstract

Tunneling dynamics and tunneling trajectories are modeled exactly by complex-extended Hamilton–Jacobi formulation in this paper. It is found that the wave-like properties of tunneling particles, such as reflection, refraction, and transmission resonance, can be identified and explained in terms of particle's motion in complex space with the tunneling time defined as the usual sense of classical time. Following the complex trajectories determined by the complex Hamilton equations of motion, we can connect classical trajectories smoothly with tunneling trajectories using position and velocity continuity at the interface of the media, locate the particle's position at any instant, and find the time spent by a particle within the potential. A microscopic tunneling model is also developed to explain the probabilistic nature why a particle with the same incident conditions sometimes transmits the potential and sometimes is reflected from the potential.

1. Quantum motion in complex spacetime

Tunneling is a pure quantum mechanical phenomenon that a particle can cross a barrier with potential V, even if its total energy E is strictly less than V. Many phenomena related to tunneling are widely observed and applied in many areas of microscopic science and technology. Nevertheless, the understanding of this phenomenon to date does not seem complete yet. There are two main challenges existing in the study of tunneling: one is the tunneling time problem and the other is the connection problem. Time is a simple classical concept but it has no exact quantum mechanical counterpart, since time enters standard quantum mechanics as a parameter, not as an observable. In the absence of a widely accepted time operator and of a unifying "clock principle" in the present quantum realm, the definition and calculation of tunneling time depends on how one sets out to measure it [1,2]. The other critical issue of tunneling is the connection problem regarding to the construction of the global solution by connecting the local solutions separated by the classical turning points at which the WKB solution becomes singular. It was pointed out [3,4] that the transition to classically inaccessible regions and the connection of dynamically separated classical paths could be realized only by including complex trajectories.

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There are many semi-classical approaches to tunneling problems [5–7]; it is nevertheless true that all the existing treatments of quantum tunneling dynamics by classical methods are subjected to various degrees of approximation. Tunneling dynamics is a manifestation of wave-particle duality, which is originated from the non-classical topology and geometry of quantum spacetime when projected into our $3+1$ Euclidean space [8–11]. By using the complex-space formulation of fractal spacetime [12–15], we demonstrate that the tunneling dynamics can be modeled exactly in the framework of complex-extended Hamilton mechanics. In this formulation, the definition of tunneling time is as simple as the usual classical time without the necessity of defining any time operator and the tunneling trajectory can be defined unambiguously in the complex plane such that the trajectories in classical regions and non-classical regions are connected smoothly.

It is a common belief that quantum mechanics has no dynamic equation of motion, although it has the equation for the propagation of probability, i.e., the Schrödinger equation. The lack of dynamic equation of motion may be one of the sources of controversies of quantum mechanics. Complex-extended Hamilton mechanics can resolve this problem by providing Hamilton equations of motion, which can be used to compute particle's quantum trajectories that are consistent with the observed quantum behavior. The application of Hamilton mechanics to the electron's quantum dynamics in hydrogen atom has been considered in [16], where it was found that the quantizations of energy, angular momentum and the action variable are all originated from the electron's complex motion governed by quantum Hamilton equations. Based on the derived electron's trajectory, it was made clear why the electron appears at some positions with large probability, while at some other positions with small probability.

Particle with energy E moving within a step potential with height $V_0 > E$ is classically prohibited, since this would lead to an imaginary momentum $p=\sqrt{2m(E-V_0)}=\mathrm{i}\sqrt{2m(V_0-E)}$ within the potential; however, this prohibition is no longer necessary, if we extend Hamilton mechanics to complex space and allows particle to possess imaginary parts of position and momentum. We will see in this paper that quantum tunneling is nothing but a classical motion emerged in complex space, which allows us to treat tunneling problems exactly by using classical Hamilton mechanics extended to complex space.

Here the position of a tunneling particle is described generally by $x = x_R + \mathrm{i}x_I \in \mathbf{C}$, with $x_R \in \mathbf{R}$ and $x_I \in \mathbf{R}$ being the real and imaginary parts of x, respectively. Analogously, its momentum is described by $p = p_R + \mathrm{i}p_I \in \mathbf{C}$. This generalization of particle's motion to complex space does not damage the accepted understanding of tunneling phenomena, since if the particle's motion is purely real, we automatically have $x_I = 0$. According to complex-extended Hamilton mechanics [13–15], the equations of motion of a tunneling particle moving under the action of a potential $V(x)$ is governed by the following complex quantum Hamiltonian:

$$H(x,p)=\frac{1}{2m}p^2+V(x)-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\ln\psi(x)}{\mathrm{d}x^2},\quad x,p\in\mathbf{C}. \tag{1.1}$$

It can be seen that in addition to the first two classical components, the complex Hamiltonian defined in Eq. (1.1) has an extra component

$$Q(x)=-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\ln\psi(x)}{\mathrm{d}x^2}, \tag{1.2}$$

called complex quantum potential, which is the origin of particle's quantum behavior. The quantum Hamilton equations derived from Eq. (1.1) read

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\partial H}{\partial p}=\frac{p}{m},\quad x(0)=x_0\in\mathbf{C}, \tag{1.3a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t}=-\frac{\partial H}{\partial x}=-\frac{\mathrm{d}}{\mathrm{d}x}\left(V(x)-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\ln\psi(x)}{\mathrm{d}x^2}\right),\quad p(0)=p_0\in\mathbf{C}. \tag{1.3b}$$

Substituting Eq. (1.3a) into Eq. (1.3b) yields the complex Newton's equation

$$m\frac{\mathrm{d}x^2}{\mathrm{d}t^2}=-\frac{\mathrm{d}V}{\mathrm{d}x}-\frac{\mathrm{d}Q}{\mathrm{d}x},\quad x\in\mathbf{C}. \tag{1.4}$$

This equation has the form of Newton's second law in the complex domain, in which the particle is subject to a complex quantum force $-\mathrm{d}Q/\mathrm{d}x$ in addition to the classical force $-\mathrm{d}V/\mathrm{d}x$. Next, we derive the energy conservation law in complex domain. Treating p as a function of x, we can recast $\mathrm{d}p/\mathrm{d}t$ into the form

$$\frac{\mathrm{d}p}{\mathrm{d}t}=\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{p}{m}\frac{\mathrm{d}p}{\mathrm{d}x}=\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}x}p^2. \tag{1.5}$$