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Exponential stability in Hopfield-type neural networks with impulses

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Abstract

This paper demonstrates that there is an exponentially stable unique equilibrium state in a Hopfield-type neural network that is subject to quite large impulses that are not too frequent. The activation functions are assumed to be globally Lipschitz continuous and unbounded. The analysis exploits an homeomorphic mapping and an appropriate Lyapunov function, and also either a geometric–arithmetic mean inequality or a Young inequality, to derive a family of easily verifiable sufficient conditions for convergence to the unique globally stable equilibrium state. These sufficiency conditions, in the norm $\|\cdot\|_p$ where $p \geqslant 1$, include those governing the network parameters and the impulse magnitude and frequency.

1. Introduction

Impulses are ubiquitous in both biological and artificial neural networks. For example, peaceful sleep may be suddenly interrupted, perhaps by a nightmare. However, even if there are abrupt changes in the neural state, a person may resume a deeper sleep provided the interruption is not too great nor too frequent. In the case of an artificial network for signal processing, faulty elements can produce sudden changes in the state voltages and thereby affect the normal transient behaviour in processing signals or information, and robust system design is important. Neural networks perceived as either continuous or discrete dynamical systems have been studied extensively, but the mathematical modelling of dynamical systems with impulses is a quite recent development [1,2,14–16,24–26,31]. In this paper, we demonstrate the exponential stability of a unique equilibrium state in a Hopfield-type neural network [17] consisting of m processing units, subjected to impulsive state displacements. Phenomena such as those mentioned above may be interpreted as large impulses affecting otherwise normal transient behaviour, but conditions for a neural network to resist impulses of significant magnitude do not appear to have been found before.

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2. Mathematical model

The network is described by

$$\frac{\mathrm{d}\boldsymbol{x}_i(t)}{\mathrm{d}t} = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + c_i, \quad t > t_0,\ t \neq t_k, \tag{2.1}$$

$$\text{subject to} \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^m, \quad \text{and} \quad \Delta x_i|_{t=t_k} = x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k^-)), \quad k = 1, 2, 3, \ldots$$

Here $x_i(t_k^+) \equiv x_i(t_k + 0)$, $x_i(t_k^-) \equiv x_i(t_k - 0)$ for $i \in \mathscr{I} = \{1, 2, \ldots, m\}$ and the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies

$$t_0 < t_1 < t_2 < \cdots < t_k \to \infty \quad \text{as } k \to \infty \quad \text{and} \quad \Delta t_k = t_k - t_{k-1} \geqslant \theta$$

for $k = 1, 2, 3, \ldots$, where the value $\theta > 0$ denotes the minimum time of interval between successive impulses. A sufficiently large value of θ ensures that impulses do not occur too frequently (see later), but $\theta \to \infty$ means that the network (2.1) becomes impulse free. We refer to Gopalsamy [14] for earlier discussion of this model formulation and its application.

The vector solution $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_m(t))^{\mathrm{T}} \in \mathbb{R}^m$ of (2.1) has components $x_i(t)$ piece-wise continuous on (t_0, β) for some $\beta > t_0$, such that $\boldsymbol{x}(t_k^+)$ and $\boldsymbol{x}(t_k^-)$ exist and $\boldsymbol{x}(t)$ is differentiable on the open intervals $(t_{k-1}, t_k) \subset (t_0, \beta)$. Further, we assume that $\boldsymbol{x}(t)$ is right continuous with $\boldsymbol{x}(t_k) = \boldsymbol{x}(t_k^+)$; the functions $I_k : \mathbb{R} \to \mathbb{R}$ that characterize the impulsive jumps are continuous; the neural parameters a_i, b_{ij}, c_i satisfy $a_i > 0, b_{ij}, c_i \in \mathbb{R}$; and the activation functions $f_j : \mathbb{R} \to \mathbb{R}$ with $f_j(0) = 0$ may be unbounded and continuous but satisfy

$$\begin{aligned} &|f_j(u) - f_j(v)| \leqslant L_j |u - v| \quad \text{for all } u, v, \in \mathbb{R}, \\ &|f_j(u)| \to \infty \quad \text{as } |u| \to \infty, \end{aligned} \tag{2.2}$$

where $L_j > 0$ for $j \in \mathscr{I}$ denotes a Lipschitz constant.

Sufficiency conditions on the neural parameters and the impulses have previously been obtained to guarantee the asymptotic convergence towards a unique equilibrium state of the network (2.1), associated with the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ [14]. However, the asymptotic stability of this equilibrium state is guaranteed only if the magnitudes of the impulses are neither large nor frequent. Although that is of course consistent with the view that a dynamical system tends to become unstable when subjected to sufficiently frequent impulses of large magnitude [4,22,30], in this paper a family of easily verifiable sufficient conditions on the neural parameters and the impulses is found to more generally guarantee the exponential convergence of the neural states towards the unique equilibrium state, and in any norm $\|\cdot\|_p$ where $p \geqslant 1$. The results obtained by applying a geometric–arithmetic mean inequality and a Young inequality to an appropriate Lyapunov function significantly enhance the earlier work, both with and without impulses.

3. Existence and uniqueness theorems

To provide for an associative memory [8,29,34,37], the network architecture is designed to not only store as many equilibrium states (or memories) as possible, but also to retrieve the relevant stored memory produced by a given external stimulus $\boldsymbol{c} = (c_1, c_2, \ldots, c_m)^{\mathrm{T}}$. Thus if a neural network is intended to solve an optimization problem, the circuit design of the network should ensure that all neural states approach a unique equilibrium state of the network [6,13,21,23,32]. The association of an equilibrium state with an external input vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_m)^{\mathrm{T}}$ avoids the possibility of convergence towards some local minimum that is a spurious equilibrium state, and not the optimal solution of the optimization problem.

To ensure that a unique equilibrium state exists, it has been customary to impose restrictions on the neural parameters and the activation functions of the network, such assuming they are bounded and Lipschitz continuous [11] – i.e.

$$g_j(0) = 0, \quad |g_j(u)| \leqslant M_j, \quad |g_j(u) - g_j(v)| \leqslant L_j |u - v|$$

for some positive constants M_j, L_j and any $u, v \in \mathbb{R}$. Stronger requirements have been adopted [3,9,10,12,18,20,27,35,36,39], where the activation functions are assumed to be bounded, continuous, monotonic and differentiable – i.e. $g_j \in C^1(\mathbb{R})$, $g_j'(u) > 0$ for $u \in \mathbb{R}$, $g_j'(0) = \sup_{u \in \mathbb{R}} g_j'(u) > 0$, $g_j(0) = 0$ and $g_j(u) \to \pm 1$ as $u \to \pm\infty$. Recent applications, in which the activation functions are either linear and piece-wise continuous or Gaussian, have indicated a number of advantages for removing the differentiability and monotonicity requirements [19,29,34,37]. And in applying a neural network to solve a certain class of optimization problems, Forti et al. [13] have used activation functions of exponential-type and diode-like, which are unbounded to suit the unbounded constraint requirement of the problems considered.