Chaos, Solitons and Fractals 88 (2016) 3–18

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Financial power laws: Empirical evidence, models, and mechanisms☆

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ARTICLE INFO

Article history:
Received 11 January 2016
Accepted 25 January 2016
Available online 2 March 2016

Keywords:
Power laws
Financial returns
Leptokurtosis
Volatility clustering
Agent-based modeling

ABSTRACT

Financial markets (share markets, foreign exchange markets and others) are all characterized by a number of universal power laws. The most prominent example is the ubiquitous finding of a robust, approximately cubic power law characterizing the distribution of large returns. A similarly robust feature is long-range dependence in volatility (i.e., hyperbolic decline of its autocorrelation function). The recent literature adds temporal scaling of trading volume and multi-scaling of higher moments of returns. Increasing awareness of these properties has recently spurred attempts at theoretical explanations of the emergence of these key characteristics form the market process. In principle, different types of dynamic processes could be responsible for these power-laws. Examples to be found in the economics literature include multiplicative stochastic processes as well as dynamic processes with multiple equilibria. Though both types of dynamics are characterized by intermittent behavior which occasionally generates large bursts of activity, they can be based on fundamentally different perceptions of the trading process. The present paper reviews both the analytical background of the power laws emerging from the above data generating mechanisms as well as pertinent models proposed in the economics literature.

1. Introduction

While research on power laws in income and wealth dates back to the nineteenth century (Pareto), the attention on power laws in financial data is relatively recent. The first ever manifestation of power laws in finance can probably be found in Mandelbrot [99] followed by Eugene Fama's elaboration (Fama [43]) published as the immediately succeeding paper in the same issue of the *Journal of Business*. This breakthrough very much dominated the discussion over the next thirty(!) years or so with an immense number of papers dedicated to providing supporting or contradicting evidence for the Paretian or Levy stable hypothesis. While the dust has settled over the last decade and the power-law behavior of large price changes now counts as one of the most pervasive findings in financial economics, it had remained the only power law under discussion in this area for quite some time.

Only recently was it joined by other candidates for Pareto-like behavior. By now well accepted within the scientific community is a second power law characterizing the temporal dependence structure of volatility. However one tries to proxy the unobservable quantity 'volatility'

☆ The author are thankful for many helpful comments by Claudio Cioffi, Dietrich Stauffer and Frank Westerhoff on earlier versions of the manuscript. Financial support from the European Union 7th Framework Programme under grant agreement No. 612955 during the final revision of this paper is also gratefully acknowledged.

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http://dx.doi.org/10.1016/j.chaos.2016.01.020
0960-0779/

(most straightforwardly via the squares or absolute values of financial returns), the autocorrelations of these entities appear to decay hyperbolically, i.e. Pareto-like. Although this feature is linked to the long known clustering of volatility in financial markets, the fact that the dependency in the fluctuations is of a long-range type had only been realized in the nineties. Credit for this observation is probably due to Ding et al. [37], published in the *Journal of Empirical Finance*. Later on, several papers by physicists emphasized the power law nature of this finding and its potential root in complex market interactions (cf. Lux [89]). The power laws in returns and in volatility seem to be intimately related: none of them was ever observed without the other and it, therefore, seems warranted to interpret them as the joint essential characteristics of financial data.

Very recently additional power laws have entered the scene: transaction volume (which is strongly correlated to volatility) also appears to be characterized by long-range dependence (although it is not clear whether volatility and volume share the same degree of long memory). Availability of high- frequency tick-by-tick data has furthermore revealed other types of power-law behavior, such as a power law for the number of trades in the New York Stock Exchange Trades and Quotes Database, cf. Plerou et al. [105]. Similar results are reported for the Japanese stock market, cf. Takayasu [118].

The plan of the remainder of this paper is the following: Section 2 gives a more formal description of the main financial power laws characterizing returns and volatility together with a survey of pertinent literature. After having set the scene, we turn to explanatory models. Section 3 deals with the so-called rational bubble model which emerged as a potential explanation of financial power laws from the standard body of rational expectations models in economics. Interestingly, this approach points to multiplicative stochastic processes as a type of data generating process with generic power-laws. This interesting property of the underlying process notwithstanding, the rational bubble model makes grossly incorrect numerical predictions about the magnitude of the exponent. In Section 4 we, therefore, turn to more recently proposed models in the behavioral finance literature. From the diversity of available approaches and models, we try to single out the basic ingredients and mechanisms leading to true or at least apparent power laws in simulated data. Section 5 attempts to draw some overall conclusions from the hitherto available body of literature on potential explanations of financial scaling laws.

2. Empirical power laws in finance

The modern literature in this area starts with Mandelbrot [99] and Fama [43], who both proposed the so-called Paretian or Levy stable distributions as statistical models for financial returns[1] (cotton futures were analyzed in Mandelbrot's paper). The theoretical appeal of this family of distributions is its stability under aggregation. At the time of publication of these papers, it had already been known for some time that a Generalized Central Limit Law holds for distributions with non-convergent (infinite) second moments: while existence of the second moment warrants convergence of sums of random variables (at least in the IID case and under weak dependence) towards the Gaussian, non-convergence of the variance implies convergence of the distribution of sums towards members of the family of Levy stable distributions. Under this perspective, the pronounced deviation of histograms for financial returns from the shape of the Normal distribution together with their apparent additivity (daily returns can be expressed as the sum of all intra-daily price changes) was interpreted as striking evidence in favor of the Levy hypothesis. The Levy distributions are characterized by an asymptotic power-law behavior of their tails with an index α (called the characteristic exponent) which implies a complementary cumulative density function of returns (denoted by *ret* in the following) which in the tails converges to:

$$Pr(|ret| > x) \approx x^{-\alpha}. \tag{1}$$

The Levy hypothesis restricts the power-law for returns to the admissible range of $\alpha \in (0, 2)$ which indicates the mentioned non-convergence of the second moment (with $\alpha < 1$ not even the mean would converge). Empirical estimates based upon the Levy model typically found α hovering around 1.7.

While this result was confirmed again and again when the parameters of the Levy laws were estimated themselves, other studies raised doubts in the validity of the Levy hypothesis by questioning the stability-under-aggregation property of these estimates (Hall et al., [60]) or pointed to apparent convergence of sample second moments (Lau et al., [77]). From the early nineties, however, it became common practice to concentrate on the tail behavior of the distribution itself and estimate its decay parameter via conditional maximum likelihood without assuming a particular distributional model (Hill [63]). The pertinent literature gradually converged to the insight of an exponent significantly larger than 2 and mostly close to 3, cf. Jansen and de Vries [69]; Lux [85] and Werner and Upper [126], among others. These results nicely agree with estimates obtained by physicists via their typical log–log regression approach (Cont et al., [33]; Gopikrishnan et al., [58]). The approximate cubic form of the power-law of returns is by now accepted as a universal feature of practically all types of financial markets (from share markets and futures to foreign exchange and precious metal markets). Note that this finding implies rejection of the time-honored Levy hypothesis as $\alpha \approx 3$ means that the decay of the outer part of the distribution is faster than allowed by this family of distributions. The Levy distributions might still be relevant for returns on venture capital and R&D investments, cf. Casault et al. [23]. It seems plausible that these types of very risky

[1] The quantity of interest in empirical research in financial economics is typically 'returns' defined as relative (or logarithmic) price changes over a certain time horizon. Research on the statistical properties of returns started with data at weekly or monthly frequencies but has moved on to high frequency data over time (daily and intra-daily data up to the highest frequencies at which all tick-by-tick changes are recorded).