Chaos, Solitons and Fractals 88 (2016) 107–118

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Pricing turbo warrants under stochastic elasticity of variance

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ARTICLE INFO

Article history:
Available online 30 December 2015

Keywords:
Turbo warrant option
Stochastic elasticity of variance
Ornstein–Uhlenbeck process
Multi-scale analysis

ABSTRACT

We consider an extended constant elasticity of variance (CEV) model in which the elasticity follows a stochastic process driven by a fast mean-reverting Ornstein–Uhlenbeck process. Then, we use the proposed model to examine a turbo warrant option, which is a type of exotic option. Based on an asymptotic analysis, we derive the partial differential equation of the leading and the corrected terms, which we use to determine the analytic formula for the turbo warrant call option. The parameter analysis using the extended CEV model provides us with a better understanding of the price structure of a turbo warrant call. Moreover, by comparing the turbo warrant call with a European vanilla call, we can examine the sensitivity of options with respect to the model parameters.

1. Introduction

Turbo warrants first emerged in Germany in late 2001, and experienced enormous growth in Northern Europe and Hong Kong. Turbo warrants are barrier options of the Down-and-Out type in which the rebate is measured by another exotic option. They are similar to a vanilla contract, but with two additional features: the contract has a low vega, signifying that the option price is less sensitive to changes in the implied volatility of the stock market; and the option is highly geared owing to the possibility of knockout. Eriksson [6] was the first to derive the pricing formula for a turbo warrant, based on the geometric Brownian motion (GBM) of the underlying asset.

It is well known that the assumption of GBM in the Black–Scholes (BS) model [1] in security markets is not well supported by empirical evidence. There are two major drawbacks to this assumption. The first shortcoming lies in the flat implied volatility, whereas, in reality, the volatility fluctuates depending upon the underlying asset price and time. The second weakness is the underestimation of extreme moves, generating a tail risk, which can be hedged with out-of-the-money options. Thus, many alternative underlying models have been proposed in order to overcome these problems and to extend the GBM.

One such alternative model is the local volatility model, in which the volatility depends upon the underlying asset itself. The most well-known local volatility model is the constant elasticity of variance (CEV) model, in which the volatility is expressed in terms of the power of the underlying asset. This model was first proposed by Cox [3] and Cox and Ross [4]. However, the CEV model is restricted in terms of delta and vega hedging, as described by Hagan et al. [10], who call it the stochastic alpha beta rho (SABR) model. The SABR model without the drift terms applies only to the calibration of short maturity options and is difficult to apply to option pricing. In addition, changes in the volatility and the underlying risky asset price are perfectly correlated in the CEV model, either positively or negatively. However, this correlation is not supported by empirical evidence. Therefore, a different alternative model is needed.

In the original CEV model, the volatility is given by $\sigma S_t^{\theta-1}$, where S_t is the underlying asset price, σ is a volatility scale

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http://dx.doi.org/10.1016/j.chaos.2015.11.043

parameter, and θ is the constant elasticity. However, Kim et al. [11] demonstrated that the elasticity parameter θ fluctuates fast around a mean level with a small amplitude (see Figure 1.1 of Kim et al. [11]). The model derived from this observation is called the "stochastic elasticity of variance" (SEV) model. Based upon this concept of stochastic elasticity of variance, Kim et al. [12] have derived an European option price formula for an asset price model which is composed of a multiscale version of the elasticity of CEV and the fast-mean reverting process and have demonstrated that the SEV model improves the markets volatility forecast driven by the CEV model. Yoon and Kim [21] studied a closed-form solution of perpetual American option prices using the SEV model. Using several parameters, they analyzed the changes in option prices and free-boundary values. Then, Yoon et al. [22] investigated perpetual American option prices and optimal exercise prices using a generalized constant elasticity of variance model, and examined the behavior of these prices with respect to the model parameters. Based on the SEV model, Yang et al. [19] derived the partial differential equations (PDEs) for the leading order and correction order terms in order to solve the portfolio optimization problem. Then, they investigated the effect of the SEV on the portfolio selection. Yoon [20] obtained pricing formulae for perpetual American options based on a multi-scale SEV. Here, the elasticity is driven by a fast mean-reverting process and a slowly varying diffusion process. Then, they observed how the parameters included in the stochastic elasticity influenced the option prices and the optimal boundary prices. In addition, Yoon et al. [23] analyzed the impact of stochastic interest rates on the pricing of European vanilla options under the SEV model.

In this study, we focus on the pricing of turbo warrants under the SEV model, and propose a model that differs from those used recently to study security prices. The pricing of turbo warrants has been studied by Wong and Chan [17] under the CEV model and a stochastic volatility model, by Wong and Lau [18] under a jump diffusion model, and by Eriksson [6] under the BS model. In fact, Wong and Chan [17] obtained analytic solutions for turbo warrants under the stochastic volatility model to examine the behavior and sensitivity of turbo warrants to implied volatility. Furthermore, Wong and Lau [18] derived analytic solutions of turbo warrants based on the double exponential jump diffusion model, using a Laplace transform to investigate the sensitivity of the turbo warrant to the jump parameters. Then, Lee et al. [14] examined the pricing of turbo warrants within the hybrid setting of stochastic and local volatility provided by Choi et al. [2]. We attempt to apply the turbo warrant option to the SEV model in order to derive the PDEs in terms of each order term, as well as to find the analytic solutions of the order terms under the SEV model. Furthermore, we compare the pricing of turbo options with that of European vanilla options and analyze the sensitivity between them against the model parameters under the SEV model.

This paper is organized as follows. Section 2 reviews the pricing of turbo warrant call options and constructs the PDEs for the price under the SEV model. In Section 3, we use an asymptotic analysis to solve the PDEs under the assumption that the SEV is driven by a fast mean-reverting process. Then, we use the proposed method to derive analytic solutions for the leading and correction terms. In Section 4, we observe the implications of the pricing formula, based on numerical calculations, and consider which parameters have the biggest impact on the SEV option price. Finally, Section 5 concludes the paper.

2. Model formulation

We define a turbo warrant contract as per Lee et al. [14]. If S_t is the underlying asset price time t, then a turbo warrant call pays the option holder $(S_T - K)^+$ at maturity T, assuming that a specified barrier $H \geq K$ has not been passed by S_t at any time prior to maturity. Let us define τ_H as the first time that the asset price hits the barrier H; that is, $\tau_H = inf\{t | S_t \leq H\}$. Under $\tau_H \leq T$, the contract is void and a new contract begins, which is a call option with a payoff given by the difference between $m_{\tau_H}^{T_0} := min_{\tau_H \leq t \leq \tau_H + T_0} S_t$ and the strike price K, with time to maturity T_0. Therefore, the turbo call contract is given by

$$\begin{aligned}\mathbf{TC}(t,s) &= E_t[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\tau_H > T\}} | S_t = s] \\ &\quad + E_t[e^{-r(\tau_H + T_0 - t)}(m_{\tau_H}^{T_0} - K)^+ \mathbf{1}_{\{\tau_H \leq T\}} | S_t = s], \quad (1)\end{aligned}$$

where E_s denotes the expectation with respect to the risk-neural probability Q^{γ} given information up to time s, that is, $E_s = E[\cdot | \mathcal{F}_s]$. From now on, we will omit the subindex s when $s = t$.

The turbo warrant call formulated by (1) is divided into two parts. The first part takes after a down-and-out barrier (**DOC**) call option with a zero rebate. The second part is a down-and-in lookback (**DIL**) call option. Thus, the following equations are satisfied:

$$\begin{aligned}\mathbf{DOC}(t,s) &= E[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\tau_H > T\}} | S_t = s], \\ \mathbf{DIL}(t,s,T_0) &= E[e^{-r(\tau_H + T_0 - t)}(m_{\tau_H}^{T_0} - K)^+ \mathbf{1}_{\{\tau_H \leq T\}} | S_t = s]. \quad (2)\end{aligned}$$

Let us define $\mathbf{LB}(\tau_H, S_{\tau_H}, T_0) = E_{\tau_H}[e^{-rT_0}(m_{\tau_H}^{T_0} - K)^+]$ as a non-standard lookback option. Then, let $\mathbf{LC}_{\mathrm{fl}}(t, s, m, T)$ be the price of the floating strike lookback call on $S_t = s$, with realized minimum m and time to maturity T. Thus, we obtain the result given in the following theorem.

Theorem 2.1. *When underlying process is continuous stochastic process, then, at $t < \tau_H$, the turbo call warrant is expressed by*

$$\begin{aligned}\mathbf{TC}(t,s) &= \mathbf{DOC}(t,s) \\ &\quad + E[e^{-r(\tau_H - t)} \mathbf{1}_{\{\tau_H \leq T\}} \mathbf{LB}(\tau_H, S_{\tau_H}, T_0) | S_t = s] \\ &= \mathbf{DOC}(t,s) \\ &\quad + E[e^{-r(\tau_H - t)} \mathbf{1}_{\{\tau_H \leq T\}} \mathbf{LB}(\tau_H, H, T_0) | S_t = s] \quad (3)\end{aligned}$$

where

$$\begin{aligned}\mathbf{LB}(\tau_H, S_{\tau_H}, T_0) &= \mathbf{LC}_{\mathrm{fl}}(S_{\tau_H}, min(S_{\tau_H}, K), T_0) \\ &\quad - \mathbf{LC}_{\mathrm{fl}}(S_{\tau_H}, S_{\tau_H}, T_0). \quad (4)\end{aligned}$$

In particular, if $S_t = H$ with $t = \tau_H$, then

$$\mathbf{TC}(\tau_H, H) = E_{\tau_H}[e^{-rT_0}(m_{\tau_H}^{T_0} - K)^+] = \mathbf{LB}(\tau_H, H, T_0).$$

Proof. Refer to Wong and Chan [17]. □

Corollary 2.1. *When underlying process follows the time-independent local volatility model (e.g. BS model and CEV*