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Alternative measure of multifractal content and its application in finance

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An alternative method for analysis of multifractal properties of time series is provided. We propose a new kind of measure of multifractality strength which takes into account the behavior of multifractal profile of the generalized Hurst exponent $h(q)$ for all moment orders q and is not limited only to the edge values of moment orders describing the scaling properties of smallest and largest fluctuations of a given signal in multifractal detrended fluctuation analysis (MFDFA). The meaning of this new measure is clarified and its performance is investigated for synthetic multifractal data and also for examples of real signals originating from stock markets. We provide also the interpretation of the alternative method following the scaling law that links together the geometric mean value of properly normalized standard q-fluctuation function $F^2(q;\ \tau)$ in MFDFA and the window length τ in which detrending of a signal is performed. We discuss in this context the influence of multifractal bias on the new measure, i.e., the influence of effects which give similar observed features as multiscaling properties however, are not generated by temporal multiscaling autocorrelation in data. It is shown that the proposed alternative measure is robust in some extend to nonstationarity in data. As a result one may avoid problems with interpretation of multifractal profile $h(q)$ encountered in many real nonstationary signals investigated in the standard way.

1. Introduction

The multifractal detrended fluctuation analysis (MFDFA) [1] since its appearance became the major technique used for studying the multifractal properties of complex systems and time series. More than thousand of papers use MFDFA in variety of problems related with complexity (see, e.g., [2–18]). The central role in this technique is played by the qth moment ($q \in \mathrm{R}$) of the fluctuation function $F(q,\ \tau)$ [1] defined as:

$$F(q,\tau) = \left\{ \frac{1}{2N} \sum_{k=1}^{2N} [\hat{F}^2(\tau,k)]^{q/2} \right\}^{1/q} \tag{1}$$

for $q \neq 0$, and

$$F_0(\tau) = \exp\left\{ \frac{1}{4N} \sum_{k=1}^{2N} \ln[\hat{F}^2(\tau,k)] \right\} \tag{2}$$

for $q = 0$, where $\hat{F}^2(\tau,k)$ is the variance of a given signal x_i ($i = 1, \ldots, N\tau$) around its local trend, i.e.,

$$\hat{F}^2(\tau,k) = \frac{1}{\tau} \sum_{j=1}^{\tau} \left\{ x_{(k-1)\tau+j} - P_{k,j} \right\}^2 \tag{3}$$

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and τ is the size of window box in which detrending is performed, while $P_{k,\,j}$ is the polynomial trend subtracted for jth data in kth window box ($k = 1, \dots, N$).

The scaling law

$$F(q, \tau) \sim \tau^{h(q)} \tag{4}$$

estimates quantitatively the multifractal properties of a signal and is crucial within MFDFA. The strength of multifractality present in data is usually measured as a spread Δh of generalized Hurst exponent profile $h(q)$:

$$\Delta h \equiv h(-Q) - h(Q) \tag{5}$$

of generalized Hurst exponent calculated at the fixed negative ($-Q$) and positive (Q) moments respectively ($Q > 0$).

In the case of stationary series, $h(q)$ is proven to be a monotonically decreasing function and therefore $\Delta h > 0$ [19]. However, many examples shown in literature for synthetic data as well as for real signals (see, e.g., [18,20,21,24]) convince that multifractal properties may lay far outside this simplest scenario and the spread Δh, defined as above, is not indicative in these cases to show quantitatively the multifractal content of data. For instance, a non-monotonic behavior of $h(q)$ profile has been observed, what leads to twisted $f(\alpha)$ multifractal spectrum [18] if the Hölder description of multifractality is adopted with a help of Legendre transform [22,23]. In nonstationary datawith periodicity, white or color noise added (see, e.g., [20,21]) one may see domains where $h(q)$ increases or decreases with q, forming local maxima in $h(q)$, or even $h(q \to -\infty) < h(q \to +\infty)$ suggesting that fluctuations of large magnitude may appear more often in this signal than small fluctuations. This "bizarre" behavior is contrary to observations in stationary data [19]. There are also examples of realistic time series where the multifractal profile, despite small and insignificant Δh spread, remarkably expands in the central part for moment orders $|q| < Q$. One may find such behavior for instance in nonlinearly transformed financial data containing extreme events (see, e.g., [24]). The similar spread Δh may occur in such cases despite different characteristics of the $h(q)$ profile is present in the center, i.e., for $|q| < Q$.

The above results convince that a standard measurement of multifractality within MFDFA, based entirely on the Δh spread may be misleading. On the other hand there are many published papers on multifractal features of financial (and other) data which analyze the multifractality so far in the standard way (see, e.g.:[13,14,25–34], and for a review [35]). Therefore we propose in this article an alternative measure of multifractal content which takes into account the simultaneous behavior of the multifractal profile $h(q)$ for all qth order moments, not limited only to the edge values $h(\mp Q) \equiv h^{\mp}$.

In the section below the notion of alternative measure is explained and its power law property is shown. Then, in the third section, features of the new measure are investigated for synthetic signals and for examples of real nonstationary financial time series. The findings are always confronted with the standard approach of MFDFA multifractal spread. The most evident remarks and conclusions reached on the basis of proposed alternative multifractal measure are summarized in the final section of this article.

2. Introducing alternative measure of multifractal strength

As already indicated, the new measure should be sensitive to behavior of the whole $h(q)$ profile in a wide range of q moments ($-Q \le q \le Q$). Therefore it should not neglect the multiscaling properties within this range of q's, contrary to the standard measure based on the spread Δh alone which takes into account just the edge values of the generalized Hurst exponents.

In the case of multifractal data with the abandoned effects of so called multifractal bias, generally discussed in [36], a natural extension of Δh can be defined as a cumulated distance between the multifractal profile $h(q)$ and the main Hurst exponent value $H \equiv h(2)$. The choice of H as a reference point is not accidental because H is the only scaling exponent value one can refer to in monofractal case.

The simplest proposal one may postulate for a new measure denoted by $\Delta_h^{(2)}$ is the integral form

$$\Delta_h^{(2)} = \frac{1}{Q}\int_{-Q}^{Q} |h(q) - h(2)|\, dq \tag{6}$$

describing the average distance between $h(q)$ profiles for respective multifractal and monofractal cases. This proposal is not unique but is distinguished in some sense because it leads to interesting scaling interpretation as we will see below. It turns out that $\Delta_h^{(2)}$ is in fact the scaling exponent in the new power law linking the geometric mean of the detrended qth order fluctuations $F(q, \tau)$ of the signal for all q values in time window of size τ and this size itself. Indeed, let us consider the qth moment of normalized detrended fluctuation $\tilde{F}(q, \tau)$ defined as follows:

$$\tilde{F}(q, \tau) = \begin{cases} \dfrac{F(q, \tau)}{\sigma_\tau} & \text{for } q \le 2 \\[2ex] \dfrac{\sigma_\tau}{F(q, \tau)} & \text{for } q > 2 \end{cases} \tag{7}$$

where σ_τ is the standard deviation of detrended fluctuation, i.e., $F(2, \tau)$ in time window of length τ. Using the power law from Eq. (4) we may write the relationship

$$\log \tilde{F}(q, \tau) = |h(q) - h(2)| \log(\tau) + C(q) \tag{8}$$

where $C(q)$ is some function – generally dependent on q but independent on τ.

Hence, using Eq. (6)

$$\int_{-Q}^{Q} \log \tilde{F}(q, \tau)\, dq = Q\Delta_h^{(2)} \log(\tau) + \int_{-Q}^{Q} C(q)\, dq \tag{9}$$

The first integral in LHS of Eq. (9) can be approximated by a discrete sum when replacing dq by $2Q/n$ and q by $-Q + k\frac{2Q}{n}$, where $k = 0, 1, \dots, n$:

$$\sum_{k=0}^{n} \log \tilde{F}\left(-Q + k\frac{2Q}{n}, \tau\right) \frac{2Q}{n} = Q \log \left[\prod_{k=0}^{n} \tilde{F}^2\left(-Q + k\frac{2Q}{n}, \tau\right)\right]^{\frac{1}{n}} \tag{10}$$