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A closed form solution for vulnerable options with Heston's stochastic volatility

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ABSTRACT

Over-the-counter stock markets in the world have been growing rapidly and vulnerability to default risks of option holders traded in the over-the-counter markets became an important issue, in particular, since the global finance crisis and Eurozone crisis. This paper studies the pricing of European-type vulnerable options when the underlying asset follows the Heston dynamics. In this paper, we obtain a closed form analytic formula of the option price as a stochastic volatility extension of the classical Heston formula and find how the stochastic volatility effect on the Black–Scholes price as well as on the decreasing speed of the option price with credit risk depends on moneyness.

1. Introduction

Over the counter (OTC) markets constitute a significant portion of the world financial markets and they are growing continuously although they are less transparent and operate with fewer rules than do exchanges. For example, about 40% of United States stock trades are made on OTC markets (cf. http://www.reuters.com/article/2014/04/06/us-dark-markets-analysis-idUSBREA3508V20140406). Unlike the transaction on exchange traded markets, the transaction on OTC markets does not guarantee the promise of payments between buyer and seller so that the possibility of credit default exists in it. In fact, all the securities and derivatives involved in the financial turmoil which was initialized by a 2007 breakdown in the US mortgage market and finally yields the global financial crisis were traded in OTC markets. So, the credit risk of financial products traded on OTC markets has become a more important issue in finance.

The holder of options traded on OTC markets is always vulnerable to default risk of the option writer. Options exposed to the default risk are called vulnerable options. Johnson and Stulz [4] initiated the pricing of European type vulnerable options under the Black–Scholes model. While they obtained a pricing formula of an option which has only counterparty liability, Klein [5] studied a option structure under which the option price depends on the correlation between the option writer's assets and the underlying asset. Also, the American style vulnerable options were studied by Klein and Yang [6].

The studies quoted above assume that the volatility of underlying asset is constant until the maturity of option. Even though the assumption of constant volatility gives us many advantages such as a closed form analytic formula, the crucial weakness is that it can't capture volatility smile or skew which is a well-known feature of the implied volatility surface. Refer to Rubinstein [9] and Jackwerth and Rubinstein [3] for details. However, relaxing the assumption of the constant volatility of the underlying price process and allowing the volatility to evolve stochastically in time, stochastic volatility models are able to capture the well-known features of the implied volatility surface. Yang et al. [11] provided an approximate formula for European

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vulnerable options with a fast mean-reverting stochastic volatility model of Fouque et al. [1]. However, the Heston model [2] among stochastic volatility models brings wide popularity because it leads to an explicit integral formula which is easy to compute for European vanilla options. So, in this paper, we obtain a closed form analytic formula for the vulnerable options under the Heston model and so it is now much easier to handle risk management problems with credit risk of options. We also investigate how the stochastic volatility effect on the option price depends upon moneyness (the relative position of the price of an underlying asset with respect to the strike price of an option) as the credit risk of the option writer increases.

The rest of paper is organized as follows. In Section 2, we formulate a partial differential equation (PDE) problem for a European vulnerable option under the Heston model. In Section 3, using the Green function and Fourier transform method, we solve the problem and obtain a closed form formula of the option price. Section 4 is devoted to investigate how the Heston volatility effect depends upon moneyness. Section 5 concludes.

2. Model formulation

We use the framework of Klein [5] for a European vulnerable call option but replace the Black–Scholes model adopted there for a given underlying asset by the Heston model in this paper.

Let S_t be the value of the underlying asset at time t satisfying the stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma_s \sqrt{Z_t} S_t dW_t^s, \tag{1}$$

$$dZ_t = \kappa(\theta - Z_t)dt + \sigma_z \sqrt{Z_t} dW_t^z \tag{2}$$

under a martingale measure, where r is the riskless expected return rate of the asset and Z_t is a process driving the mean-reverting volatility of the asset. Here, σ_s (the coefficient of the volatility of the return), κ (the mean-reversion speed of the variance process Z_t), θ (the mean level of the variance process) and σ_z (the coefficient of the volatility of the variance process) are assumed to be constants.

Let V_t denote the market value of the assets that the option writer posses at time t satisfying the SDE

$$dV_t = rV_t dt + \sigma_v \sqrt{Z_t} V_t dW_t^v \tag{3}$$

under the martingale measure, where σ_v (the volatility of the assets) is a constant. The correlation structure of the Brownian motions involved in the model above is given by

$$d\langle W_t^s, W_t^z\rangle = \rho_{sz} dt,$$
$$d\langle W_t^s, W_t^v\rangle = \rho_{sv} dt,$$
$$d\langle W_t^z, W_t^v\rangle = \rho_{zv} dt.$$

As in Klein [5], let payoff function of the vulnerable option be given by

$$h(S_T, V_T) = (S_T - K)^+ \left(1_{\{V_T \geq D^*\}} + 1_{\{V_T < D^*\}} \frac{(1-\alpha)V_T}{D} \right), \tag{4}$$

where K is the strike price of the option, α is the dead-weight costs of the financial stress expressed as a percentage of the value of the assets of the option writer, D^* is the value of the other liabilities of the option writer, and D denotes a threshold value which may be larger than D^* because of the possibility that counterparty keeps operation even while V_T is less than D^*. If V_T is less than the constant default boundary D^*, a credit loss occurs and subsequently the option writer pays out only the proportion $\frac{(1-\alpha)V_T}{D}$ of the nominal claim.

From the Feynman–Kac formula (cf. [8]), the option price $P(t, s, v, z)$ defined by

$$P(t,s,v,z) := \mathbf{E}^* \left[e^{-r(T-t)} (S_T - K)^+ \left(1_{\{V_T \geq D^*\}} + 1_{\{V_T < D^*\}} \frac{(1-\alpha)V_T}{D} \right) \middle| S_t = s, V_t = v, Z_t = z \right]$$

satisfies a partial differential equation (PDE) given by

$$\frac{\partial P}{\partial t} + rs\frac{\partial P}{\partial s} + \frac{1}{2}\sigma_s^2 z s^2 \frac{\partial^2 P}{\partial s^2} + \kappa(\theta - z)\frac{\partial}{\partial z} + \frac{1}{2}\sigma_z^2 z \frac{\partial^2 P}{\partial z^2} + rv\frac{\partial P}{\partial v} + \frac{1}{2}\sigma_v^2 z v^2 \frac{\partial^2 P}{\partial v^2} + \rho_{sz}\sigma_z\sigma_s z s \frac{\partial^2 P}{\partial s \partial z} + \rho_{sv}\sigma_s\sigma_v s z v \frac{\partial^2 P}{\partial s \partial v} + \rho_{zv}\sigma_z\sigma_v z v \frac{\partial^2 P}{\partial z \partial v} - rP = 0 \tag{5}$$

with the boundary condition $P(T,s,v,z) = (s-K)^+ (1_{\{v \geq D^*\}} + 1_{\{v < D^*\}} \frac{(1-\alpha)v}{D}) = h(s,v)$.

3. Option price formula

In this section, we solve the PDE (5) by using the Green function and Fourier transform method. For convenience, we first define the operator $\mathcal{L}_H$ by

$$\mathcal{L}_H := \frac{\partial}{\partial t} - r + r\left(s\frac{\partial}{\partial s} + v\frac{\partial}{\partial v} \right) + \frac{1}{2} z \left(\sigma_s^2 s^2 \frac{\partial^2}{\partial s^2} + \sigma_v^2 v^2 \frac{\partial^2}{\partial v^2} + {\sigma_z}^2 \frac{\partial^2}{\partial z^2} \right) + \kappa(\theta - z)\frac{\partial}{\partial z} + z\left(\rho_{sz}\sigma_s\sigma_z s \frac{\partial^2}{\partial z \partial s} + \rho_{sv}\sigma_s\sigma_v s v \frac{\partial^2}{\partial v \partial s} + \rho_{vz}\sigma_v\sigma_z v \frac{\partial^2}{\partial z \partial v} \right).$$

Then the PDE problem for the option price $P(t, s, z, v)$ is expressed by

$$\mathcal{L}_H P(t,s,v,z) = 0, \quad t < T, \quad P(T,s,v,z) = h(s,v). \tag{6}$$

To obtain a solution of the problem (6), we use the change of variables, the Green function method and the Fourier transform method sequentially as follows. First, we use the following change of independent and dependent variables.

$$\tau = T - t, \quad p = r(T-t) + \log s, \quad q = r(T-t) + \log v,$$
$$P(t,s,v,z) = P'(\tau, p, q, z) e^{-r\tau}.$$

Then the PDE problem (6) becomes a problem for $P'(\tau, p, q, z)$ as follows.