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Generation and circuit implementation of fractional-order multi-scroll attractors

Liping Chen[a,*], Wei Pan[a], Ranchao Wu[b], Kunpeng Wang[c,d], Yigang He[a]

[a] School of Electrical Engineering and Automation, Hefei University of Technology, Hefei 230009, China
[b] School of Mathematics, Anhui University, Hefei 230039, China
[c] School of Information and Engineering, Southwest University of Science and Technology, Mianyang 621010, China
[d] Robot Technology Used for Special Environment Key Laboratory of Sichuan Province, Mianyang 621010, China

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ABSTRACT

This paper considers the generating of multi-scroll chaotic attractors for a new fractional-order linear system by using the piecewise-linear function. Multi-scroll chaotic attractors are generated by extending the number of saddle equilibrium points with index 2. Poincaré map and maximum Lyapunov exponents are applied to verifying the chaotic behaviors of the generated multi-scroll chaotic attractors. A circuit for the multi-scroll attractor is designed and simulated. Moreover, physical experiment of 3-scroll attractors and 5-scroll attractors are implemented. The numerical simulation, the circuit simulation and hardware experimental results are in accordance with each other, which verifies the effectiveness and physical realization of the approach.

1. Introduction

Fractional calculus is an old branch of mathematics, which mainly deals with derivatives and integrals of arbitrary non-integer order. It was firstly introduced 300 years ago, but it only developed as a pure mathematical branch [1–4]. In the last few decades, it is found that the fractional-order derivatives are extremely useful to describe many real-world phenomena in fields such as acoustics and thermal systems, rheology, material and mechanical systems, signal processing and system identification, reconfigurable hardwares and so on [5–8]. Along with the development of fractional calculus, it was proven that many fractional-order nonlinear differential systems behave chaotically, for instance, fractional-order Duffing oscillators [9], fractional-order Chua circuit system [10], fractional-order Rössler system [11], fractional-order Chen system [12], fractional-order Lorenz system [13], fractional-order Lü system [14], fractional order Liu system [15] and so on. Due to the nonlocal properties of fractional differential operators, topological structures of fractional-order systems are different from the traditional classical differential ones. Recently, control, synchronization and circuit implementation of fractional-order chaotic systems have received much attractions.

As we all know, compared with the 3-D single-scroll chaotic attractors, 3-D multi-scroll chaotic attractors show more complex dynamic behaviors, which indicate that multi-scroll chaotic attractors have a general potential applications in communications, cryptography and many other fields. There have been a large number of works devoted to the research of circuit designs for generating multi-scroll chaotic attractors, see [16–25] and the references therein. However, most of the aforementioned multi-scroll chaotic attractors were generated by traditional Chua circuit or the integer-order linear systems. Recently, the generation of multi-scroll attractors of fractional differential systems has become a rising concern. For

* Corresponding author. Tel.: +86 18326658916.
E-mail address: lip_chenhut@126.com, lip_chen@hfut.edu.cn (L. Chen).

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instances, by using a simple stair function and hysteresis series switching function, the one-directional (1-D) n-grid scroll, two-directional (2-D) ($n \times m$)-grid scroll and three-directional (3-D) ($n \times m \times l$)-grid scroll attractors are created from a fractional linear autonomous system respectively in [26] and [27]. Ahmad introduced a step function method for creating n-scroll attractors from fractional order systems [28]. Ref. [29] introduced a multi-scroll chaotic attractor which is generated by switched fractional systems. Ref. [30] investigated the generation of multi-wing chaotic attractors using integer and fractional-order linear differential equation systems with switching controls.

It should be noticed that these aforementioned multi-scroll chaotic attractors from fractional-order system were only verified by numerical simulations [26–30]. However, it is much more difficult to generate multi-scroll chaotic attractors by physical electronic circuits. To the best of our knowledge, there are few results about circuit realization of multi-scroll chaotic attractors generated from fractional-order systems. Motivated by the above discussions, the aim of the paper is to propose a new systematic design approach for generating multi-scroll attractors based on fractional-order linear systems by employing the stair function, the numerical simulation and the circuit implementation are both given to verify our method.

The paper is organized as follows. In Section 2, the fractional derivative and some lemmas are introduced. An approach for creating the multi-scroll attractors from the fractional-order linear system is proposed in Section 3. Circuit implementation of the chaotic attractors is presented in Section 4. Conclusions are finally drawn in Section 5.

2. Preliminaries

Fractional calculus is a generalization of integration and differentiation to noninteger-order fundamental operator. There are many different kinds of definitions for fractional derivatives, but the most commonly used ones are the Riemann–Liouville definition and the Caputo definition. Since Caputo definition has a wider range of application in engineering. Here, Caputo fractional derivative operator D^{α} is adopted.

Definition 1 [3]**.** The fractional integral (Riemann–Liouville integral) $D^{-\alpha}$ with fractional order $\alpha \in R^{+}$ of function $x(t)$ is defined as

$$D^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau,$$

where $\Gamma(\cdot)$ is the gamma function, $\Gamma(\tau) = \int_0^{\infty} t^{\tau-1}e^{-t}dt$.

Definition 2 [3]**.** The Caputo derivative of fractional order α of function $x(t)$ is defined as follows

$$D^{\alpha}x(t) = D^{-(n-\alpha)}\frac{d^n}{dt^n}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau,$$

where $n-1 \le \alpha < n \in Z^{+}$.

The stability of fractional linear system is quite different from the integer case. So in order to obtain the main results, the following definitions and lemmas are presented here first.

Definition 3 [26]**.** Consider a general n-dimensional fractional system

$$D^{\alpha}(X) = f(X), \tag{1}$$

the roots of the equation $f(X) = 0$ are called the equilibrium points of fractional differential system, where $D^{\alpha}(X) = (D^{\alpha}x_1, D^{\alpha}x_2, ..., D^{\alpha}x_n)^T$, $X = (x_1, x_2, ..., x_n)^T \in R^n$.

Lemma 1 *[3]***.** *For $n = 3$, system (1) is asymptotically stable at the equilibrium point O if $|\arg(\lambda_i(J))| > \alpha\pi/2$, $i = 1, 2, 3$. J denotes the Jacobi matrix of f(X), λ_i are the eigenvalues of J.*

Lemma 2 *[31]***.** *The equilibrium point O of system (1) is unstable if the order α satisfies the condition below for at least one eigenvalue:*

$$\alpha > \frac{2}{\pi}\arctan\frac{|Im(\lambda)|}{|Re(\lambda)|}. \tag{2}$$

where $Im(\lambda)$ and $Re(\lambda)$ denote real and imaginary part of λ respectively.

Lemma 3 *[27]***.** *For $n = 3$, if one of the eigenvalues $\lambda_1 < 0$ and the other two conjugate eigenvalues $|\arg(\lambda_2)| = |\arg(\lambda_2)| < \alpha\pi/2$, then the equilibrium point O is called a saddle point with index 2; if one of the eigenvalues $\lambda_1 > 0$ and the other two conjugate eigenvalues $|\arg(\lambda_2)| = |\arg(\lambda_3)| > \alpha\pi/2$, then the equilibrium point O is called a saddle point with index 1.*

3. Generating multi-scroll attractors

In this section, we will discuss the generating of multi-scroll chaotic attractors from a three-dimensional fractional-order linear system via a stair function controller. In 2000, Sprott proposed the following Jerk system [32], which is described as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -x - y - \beta z + G(x), \end{cases} \tag{3}$$

where x, y, z are dimensionless variable, β is a suitable constant, $G(x)$ is one of a number of elementary piecewise linear functions. When $\beta = 0.5$, $G(x) = sgn(x)$, characteristic roots of Jacobian matrix at equilibrium point (0, 0, 0) are -0.8038, $0.1519 + 1.1050i$ and $0.1519 - 1.1050i$, so (0, 0, 0) is saddle-focus point of index 2. When initial conditions are chosen as (0, 1, 0), Lyapunov exponents of system (3) are 0.152, 0 and -0.652 [33]. 2-scroll chaotic attractors are displayed in Fig. 1.

Now, we consider the following fractional-order linear system derived from system (3) with $G(x) = 0$

$$\begin{cases} D^{\alpha}x = y, \\ D^{\alpha}y = z, \\ D^{\alpha}z = -x - y - \beta z. \end{cases} \tag{4}$$

where α is order and $0 < \alpha < 1$.

To guide the linear system (4) to generate chaotic behavior, it needs to add a nonlinear controller to stretch and fold the trajectories of the system repeatedly. To this end,